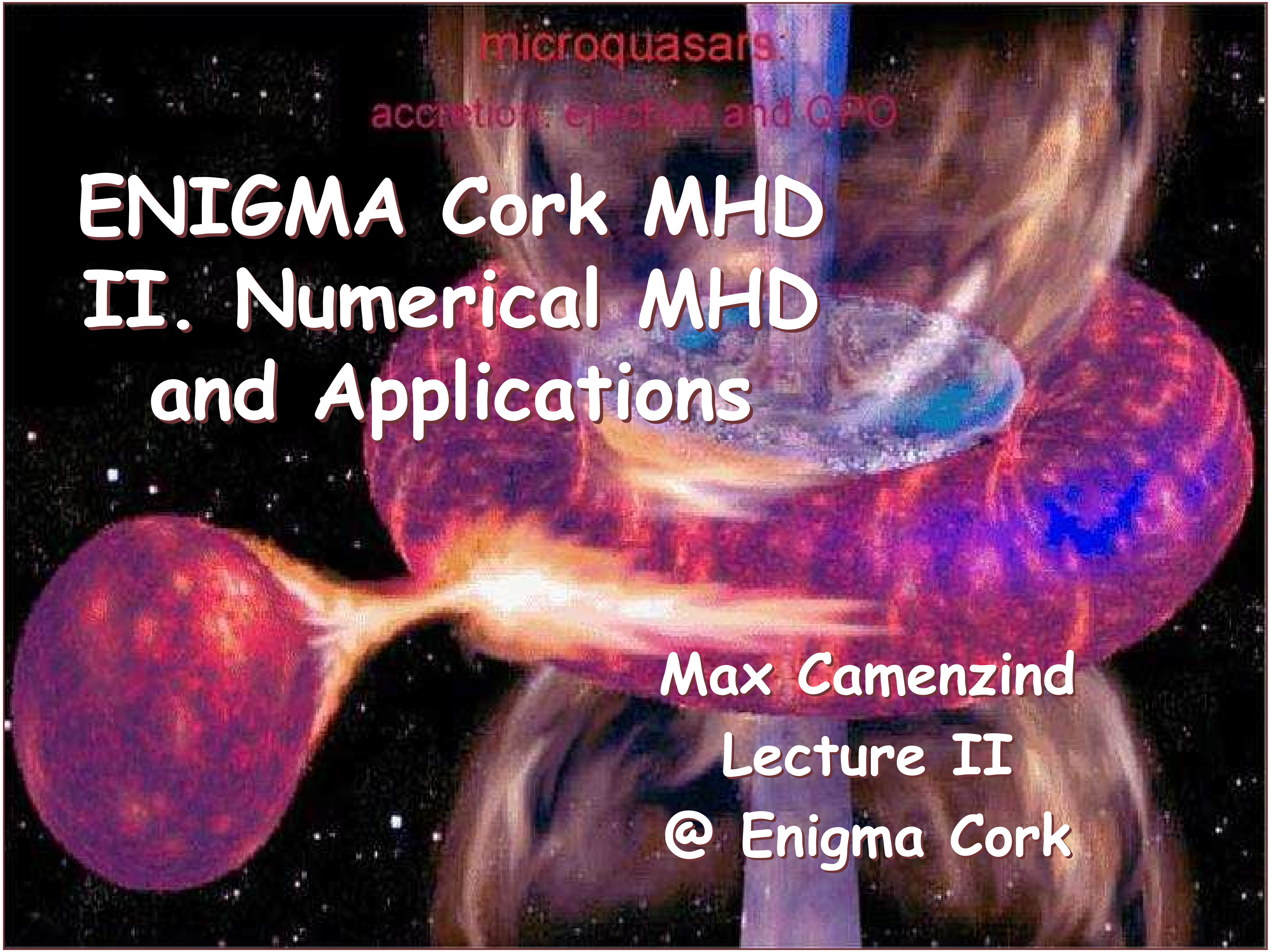

ENIGMA Cork MHD II. Numerical MHD and Applications

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Lecture II

@ Enigma Cork

Topics of Numerical MHD

- Basic Equations: MHD Primer and Outlook to Relativistic MHD
- Intermezzo II: Characteristic Waves and Turbulence, characteristic speeds
- Div(B)-Problem
- Numerical Schemes
- Application I: Disk simulations
- Application II: Jet simulations

MHD Primer

- Magnetohydrodynamics (MHD) equations describe flows of conducting fluids (ionized gases, liquid metals, plasma) in presence of magnetic fields.
- Lorentz forces act on charged particles and change their momentum and energy. In return, particles alter strength and topology of magnetic fields.
- Range of validity of MHD equations, especially of ideal MHD is narrow. Therefore, very few physical systems are truly ideal MHD !

- MHD equations are typically derived under following assumptions:
  - Fluid approximation (often, single-fluid approximation);
  - Charge neutrality: $\rho_e \sim 0$;
  - Simple transport coefficients;
  - No relativistic effects;
  - In ideal MHD: infinite conductivity (zero resistivity), zero viscosity and zero thermal diffusivity.

Ideal MHD Equations (1 component)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{V}) = 0$$

$$\rho\left(\frac{\partial \mathbf{V}}{\partial t} + (\mathbf{V} \cdot \nabla)\mathbf{V}\right) = -\nabla p + \mathbf{j} \times \mathbf{B}$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\rho \mathbf{V} H - (\mathbf{B} \times (\mathbf{V} \times \mathbf{B}))) = 0$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \times (\mathbf{V} \times \mathbf{B}) = 0$$

$$\nabla \cdot \mathbf{B} = 0$$

ρ: mass density
p: pressure
V: velocity
B: magnetic field
j: current density
H: enthalpy
ρH = e + p
E: total energy

Prominent MHD Solvers

- ZEUS3D (Stone & Norman 1992)
- NIRVANA2 (AMR, not parallel, U. Ziegler 1997)
- NIRVANA_CP (vectorized, cooling, parallel, LSW)
- NIRVANA3 (Cartesian, not parallel, U. Ziegler 04)
- **FLASH** (parallel, AMR, not spherical, in progress)
- VAC (parallel, AMR; Rony Keppens)
- PLUTO (Chicago solar physics group, Torino)
- BATSRUS (AMR, parallel, Cartesian, developped by Ken Powell for „space weather")
- All these codes will be „benchmarked" by JETSET.

Advective Form of MHD Equations (ZEUS3D, NIRVANA)

$$
\begin{aligned}
\partial_{\rm t}\rho &= -\nabla\cdot(\rho\vec{v}) \\
\partial_{\rm t}(\rho\vec{v}) &= -\nabla\cdot(\rho\vec{v}\otimes\vec{v}) - \nabla P + \frac{1}{8\pi}\nabla\vec{B}^2 + \frac{1}{4\pi}(\vec{B}\cdot\nabla)\vec{B} - \rho\nabla\Phi \\
\partial_{\rm t} e &= -\nabla\cdot(e\vec{v}) - P\nabla\cdot\vec{v} + \frac{\eta}{16\pi^2}|\nabla\times\vec{B}|^2 + \nabla\cdot(\kappa\nabla T) + \sigma : \nabla\vec{v} \\
\partial_{\rm t}\vec{B} &= \mp\nabla\times\left(\vec{v}\times\vec{B} - \frac{\eta}{4\pi}\nabla\times\vec{B}\right) \\
\sigma_{\rm ik} &= -l_{\rm T}^2\rho\,\min(0,\nabla\cdot\vec{v})\times\left(\nabla\vec{v}_{\rm ik} - \frac{1}{3}\nabla\cdot\vec{v}\right) \\
&\quad + \; l_{\rm A}\rho\left(\delta x_{\rm i}\times\min(0,\nabla v_{\rm ik})^2\right)\delta_{\rm ik}
\end{aligned}
$$

e: internal energy density

→artificial viscosity

Basic Equations in Conservative Form (FLASH, VAC,…)

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \boldsymbol{v}) = 0$$

$$\frac{\partial \rho \boldsymbol{v}}{\partial t} + \nabla \cdot (\boldsymbol{v} \rho \boldsymbol{v} - \boldsymbol{B}\boldsymbol{B}) + \nabla p_{tot} = 0$$

$$\frac{\partial e}{\partial t} + \nabla \cdot (\boldsymbol{v} e + \boldsymbol{v} p_{tot} - \boldsymbol{B}\boldsymbol{B} \cdot \boldsymbol{v} - \boldsymbol{B} \times \eta \boldsymbol{J}) = 0$$

$$\frac{\partial \boldsymbol{B}}{\partial t} + \nabla \cdot (\boldsymbol{v}\boldsymbol{B} - \boldsymbol{B}\boldsymbol{v}) + \nabla \times (\eta \boldsymbol{J}) = 0$$

Poynting flux only appears in energy equation, but not in momentum equations.

Conservative MHD (New Approach)

- Conservation equations for
- (i) mass density ρ
- (ii) momentum density $\rho\mathbf{V}$
- (iii) total energy density $e = \rho E$
- (iv) magnetic field B, or magnetic flux
- (v) total pressure $p_{tot} = p + B^2/8\pi$,
  ➔ gas pressure

$$p = (\gamma - 1)(e - \frac{1}{2}\rho \boldsymbol{v}^2 - \frac{1}{2}\boldsymbol{B}^2)$$

Ideal MHD Equations

8D State Vector U

$$\frac{\partial}{\partial t}\begin{pmatrix}\rho\\ \rho\mathbf{V}\\ \rho E\\ \mathbf{B}\end{pmatrix}+\nabla\cdot\begin{pmatrix}\rho\mathbf{V}\\ \rho\mathbf{V}\mathbf{V}+(p+\frac{B^2}{2})\bar{\bar{\mathbf{I}}}-\mathbf{B}\mathbf{B}\\ \mathbf{V}(\rho E+p+\frac{B^2}{2})-\mathbf{B}(\mathbf{V}\cdot\mathbf{B})\\ \mathbf{V}\mathbf{B}-\mathbf{B}\mathbf{V}\end{pmatrix}=0$$

Flux Vector F

Major Properties:

- MHD equations form a hyperbolic system → Seven families of waves
- → entropy (contact), Alfvén and fast and slow magnetoacoustic waves.
- Suitable for Godunov methods

On Conservative Schemes

- We define a state vector **U** in 8D for MHD
- Time evolution given in flux conservation form ➔ change of **U** in a volume given by fluxes through the surfaces (+ volume source terms, in general).
- with initial conditions **U**(0,**x**) = $\mathbf{U}_0(\mathbf{x})$
- and boundary conditions **U**(t,$\mathbf{x}_{bc}$) = $\mathbf{U}_{bc}$(t).
- ➔ Modern solvers based on such schemes.

$$\mathbf{U} = \{\rho, \rho\vec{v}, \rho E, \vec{B}\}^{\mathrm{T}}$$

$$\frac{\partial \mathbf{U}}{\partial t} + \nabla \cdot \mathcal{F}[\mathbf{U}] = 0\,.$$

Intermezzo I: Relativistic MHD is Similar to Conservative Newtonian MHD

- Primitive Variables:
  $\mathbf{V} = (\rho, V_x, V_y, V_z, p, B_x, B_y, B_z)$
- In conserved quantities D, S_x, S_y, S_z, τ this is written similar to non-relativistic equations

$$\partial_t \mathbf{U} + \partial_x \mathbf{F}^x + \partial_y \mathbf{F}^y + \partial_z \mathbf{F}^z = \mathbf{Q} \ ,$$

State Vector Conservative Form

$$\mathbf{U} = \begin{pmatrix} D \\ S^x \\ S^y \\ S^z \\ \tau \\ B^x \\ B^y \\ B^z \end{pmatrix} \equiv \begin{pmatrix} \rho W \\ \rho h^* W^2 v^x - b^0 b^x \\ \rho h^* W^2 v^y - b^0 b^y \\ \rho h^* W^2 v^z - b^0 b^z \\ \rho h^* W^2 - p^* - b^0 b^0 - \rho W \\ B^x \\ B^y \\ B^z \end{pmatrix} .$$

State vector depends non-linearly on the primitive variables ! b is the magnetic field in the plasma frame, B in the lab-frame.

Fluxes

The fluxes, $\mathbf{F}^i$, are then

$$\mathbf{F}^i = \begin{pmatrix} \rho W v^i \\ \rho h^* W^2 v^i v^x + p^* \delta^i_x - b^i b^x \\ \rho h^* W^2 v^i v^y + p^* \delta^i_y - b^i b^y \\ \rho h^* W^2 v^i v^z + p^* \delta^i_z - b^i b^z \\ \rho h^* W^2 v^i - b^0 b^i - \rho W v^i \\ v^i B^x - B^i v^x \\ v^i B^y - B^i v^y \\ v^i B^z - B^i v^z \end{pmatrix} .$$

MHD Equations in FLASH

$$\frac{\partial \rho}{\partial t} + \nabla \cdot (\rho \mathbf{v}) = 0$$

$$\frac{\partial \rho \mathbf{v}}{\partial t} + \nabla \cdot (\rho \mathbf{v}\mathbf{v} - \mathbf{B}\mathbf{B}) + \nabla p_* = \rho \mathbf{g} + \nabla \cdot \tau$$

$$\frac{\partial \rho E}{\partial t} + \nabla \cdot (\mathbf{v}(\rho E + p_*) - \mathbf{B}(\mathbf{v} \cdot \mathbf{B})) = \rho \mathbf{g} \cdot \mathbf{v} + \nabla \cdot (\mathbf{v} \cdot \tau + \sigma \nabla T) + \nabla \cdot (\mathbf{B} \times (\eta \nabla \times \mathbf{B}))$$

$$\frac{\partial \mathbf{B}}{\partial t} + \nabla \cdot (\mathbf{v}\mathbf{B} - \mathbf{B}\mathbf{v}) = -\nabla \times (\eta \nabla \times \mathbf{B})$$

where

$$p_* = p + \frac{B^2}{2},$$

$$E = \frac{1}{2} v^2 + \epsilon + \frac{1}{2} \frac{B^2}{\rho},$$

$$\tau = \mu \left((\nabla \mathbf{v}) + (\nabla \mathbf{v})^T - \frac{2}{3} (\nabla \cdot \mathbf{v}) \right)$$

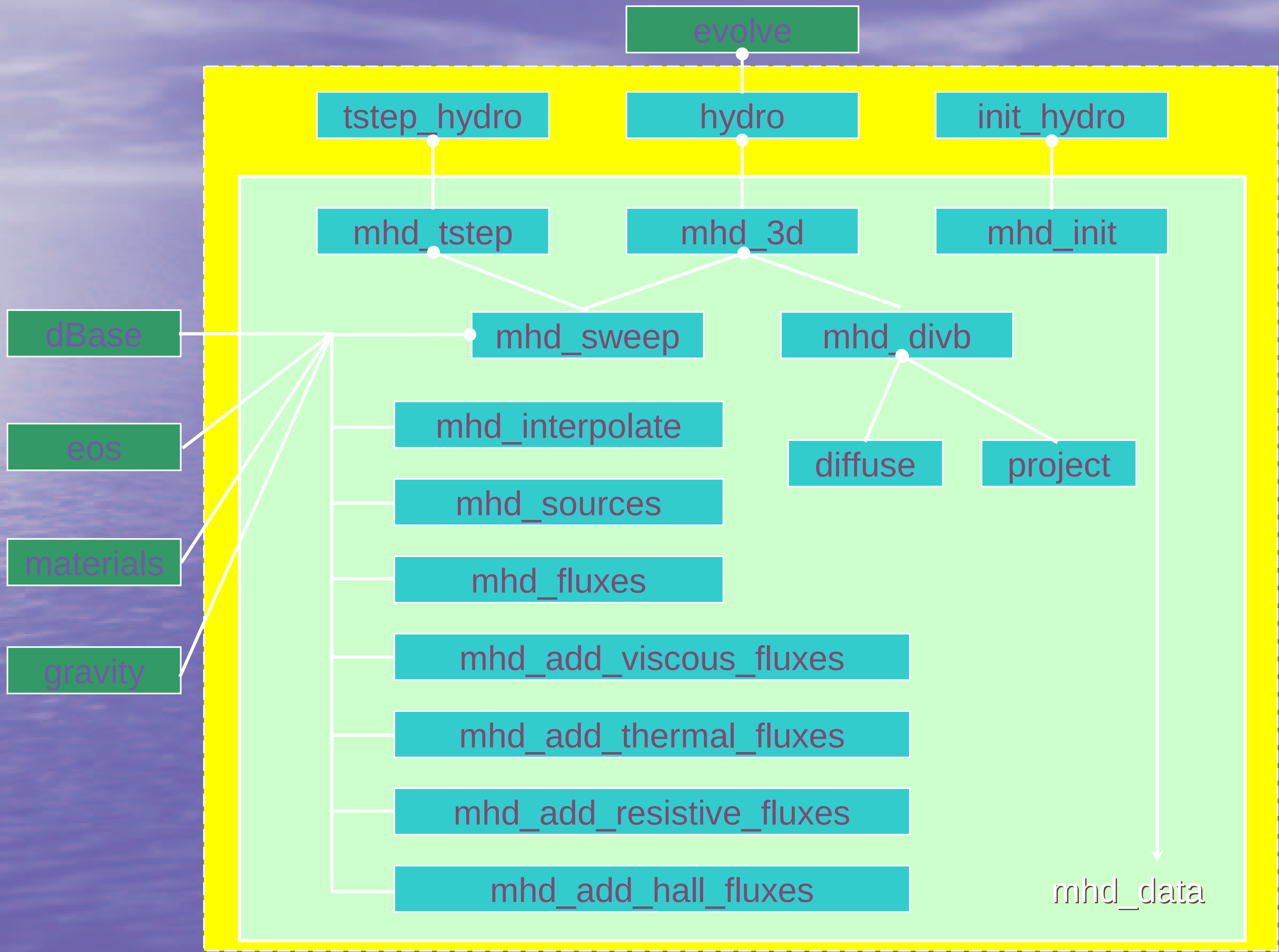
- Advective terms are discretized using slope-limited TVD scheme (E.F. Toro: Riemann Solvers ...).
- Diffusive terms are discretized using central finite differences.

FLASH MHD Config-File

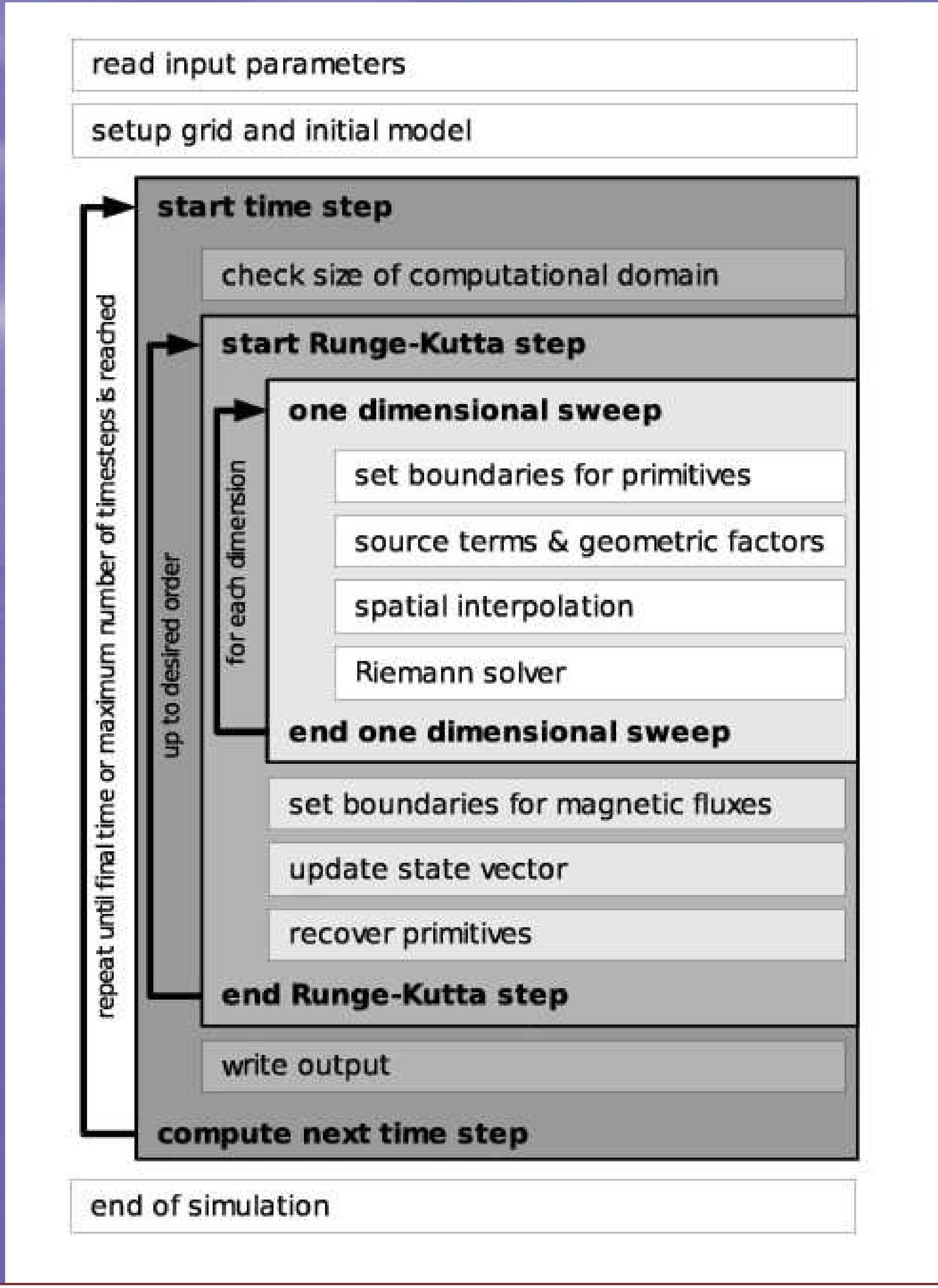
State Vector

```
# Required Modules

REQUIRES driver
REQUIRES materials/eos
REQUIRES materials/viscosity
REQUIRES materials/conductivity
REQUIRES materials/magnetic_resistivity

DEFAULT   divb_diffuse
EXCLUSIVE divb_diffuse divb_project

# Required Variables

VARIABLE dens   ADVECT NORENORM   CONSERVE # density
VARIABLE velx   ADVECT NORENORM NOCONSERVE # x-velocity
VARIABLE vely   ADVECT NORENORM NOCONSERVE # y-velocity
VARIABLE velz   ADVECT NORENORM NOCONSERVE # z-velocity
VARIABLE pres   ADVECT NORENORM NOCONSERVE # pressure
VARIABLE ener   ADVECT NORENORM NOCONSERVE # specific total energy
VARIABLE gamc NOADVECT NORENORM NOCONSERVE # sound-speed gamma
VARIABLE magx   ADVECT NORENORM   CONSERVE # x-magnetic field
VARIABLE magy   ADVECT NORENORM   CONSERVE # y-magnetic field
VARIABLE magz   ADVECT NORENORM   CONSERVE # z-magnetic field
VARIABLE divb NOADVECT NORENORM NOCONSERVE # divergence of B
VARIABLE temp NOADVECT NORENORM NOCONSERVE # temperature
VARIABLE eint NOADVECT NORENORM NOCONSERVE # specific internal energy

GUARDCELLS 2

# MHD Parameters

PARAMETER cfl          REAL       1.0    # CFL condition
PARAMETER UnitSystem   STRING    "none"  # Unit system (SI/cgs/none)
PARAMETER killdivb     BOOLEAN    TRUE   # Enable/disable DivB cleaning
PARAMETER resistive_mhd BOOLEAN   FALSE  # Turn on/off resistive terms
```

FLASH MHD Modules



MHD Flow Diagram (Operator-Split Method)



Int II: MHD Characteristic Speeds

We assume ***stationary*** ideal ***homogeneous*** conditions as the intial state of the single-fluid plasma, with vanishing average electric and velocity fields, overal ***pressure equilibrium*** and no magnetic stresses. These assumptions yield:

$$
\begin{aligned}
\mathbf{v}_0 &= 0 \\
\mathbf{E}_0 &= 0 \\
\nabla\left(p_0 + B_0^2/2\mu_0\right) &= 0 \\
\left(\mathbf{B}_0\cdot\nabla\right)\mathbf{B}_0 &= 0
\end{aligned}
$$

These fields are decomposed as sums of their background initial values and space- and time-dependent ***fluctuations*** as follows:

$$
\begin{aligned}
n &= n_0 + \delta n \\
\mathbf{v} &= \delta\mathbf{v} \\
\mathbf{E} &= \delta\mathbf{E} \\
\mathbf{B} &= \mathbf{B}_0 + \delta\mathbf{B}
\end{aligned}
$$

Linear Perturbation Theory

Because the MHD equations are nonlinear (advection term and pressure/stress tensor), the fluctuations must be small.
➔ ***Arrive at a uniform set of linear equations, giving the dispersion relation for the eigenmodes of the plasma.***
➔ ***Then all variables can be expressed by one, say the magnetic field.***

Usually, in space plasma the background magnetic field is sufficiently strong (e.g., a planetary dipole field), so that one can assume the fluctuation obeys:

$$|\delta \mathbf{B}| \ll B_0$$

In the uniform plasma with straight field lines, the field provides the only ***symmetry axis*** which may be chosen as *z*-axis of the coordinate system such that: $\mathbf{B}_0 = B_0 \hat{\mathbf{e}}_{||}$.

Linearized MHD equations I

Linarization of the MHD equations leads to three equations for the three fluctuations, δn, $\delta \mathbf{v}$, and $\delta \mathbf{B}$:

$$\frac{\partial \delta n}{\partial t} + n_0 \nabla \cdot \delta \mathbf{v} = 0$$

$$m_i n_0 \frac{\partial \delta \mathbf{v}}{\partial t} = -\nabla \left(\delta p + \frac{1}{\mu_0} \mathrm{B}_0 \cdot \delta \mathrm{B} \right) + \frac{1}{\mu_0} \left(\mathrm{B}_0 \cdot \nabla \right) \delta \mathrm{B}$$

$$\frac{\partial \delta \mathrm{B}}{\partial t} = \left(\mathrm{B}_0 \cdot \nabla \right) \delta \mathrm{B} - \mathrm{B}_0 \left(\nabla \cdot \delta \mathbf{v} \right)$$

Using the adabatic pressure law, and the derived sound speed, $c_s^2 = p_0/m_i n_0$, leads to an equation for δp and gives:

$$\frac{\partial \delta p}{\partial t} = m_i c_s^2 \frac{\partial \delta n}{\partial t} = -m_i n_0 c_s^2 \nabla \cdot \delta \mathbf{v}$$

Linearized MHD equations II

Inserting the continuity and pressure equations, and using the Alfvén velocity, $\mathbf{v_A=B_0/(\mu_0 n m_i)^{1/2}}$, two coupled vector equations result:

$$\frac{\partial \delta \mathbf{v}}{\partial t} = v_A^2 \nabla_{\|} \left(\frac{\delta \mathbf{B}_{\perp}}{B_0} \right) - \nabla \left(\frac{\delta p}{m_i n_0} \right)$$

$$\frac{\partial}{\partial t} \left(\frac{\delta \mathbf{B}}{B_0} \right) = \nabla_{\|} \delta \mathbf{v}_{\perp} - \hat{\mathbf{e}}_{\|} \left(\nabla_{\perp} \cdot \delta \mathbf{v}_{\perp} \right)$$

Time differentiation of the first and insertion of the second equation yields a second-order wave equation which can be solved by ***Fourier transformation***.

$$\begin{aligned} \frac{\partial^2 \delta \mathbf{v}}{\partial t^2} &= c_{ms}^2 \nabla \left(\nabla \cdot \delta \mathbf{v} \right) \\ &+ v_A^2 \left(\nabla_{\|}^2 \delta \mathbf{v} - \nabla \nabla_{\|} \delta v_{\|} - \hat{\mathbf{e}}_{\|} \nabla_{\|} \nabla \cdot \delta \mathbf{v} \right) \end{aligned}$$

Dispersion Relation

The ansatz of ***travelling plane waves***,

$$\delta\mathbf{v} = \delta\mathbf{v}_0 \exp(i\mathbf{k}\cdot\mathbf{x} - i\omega t)$$

with arbitrary constant amplitude, $\delta\mathbf{v}_0$, leads to the system,

$$\left[\left(\omega^2 - k_{\|}^2 v_A^2\right)\mathbf{I} - c_{ms}^2\mathbf{k}\mathbf{k} + \left(\mathbf{k}\hat{\mathbf{e}}_{\|} + \hat{\mathbf{e}}_{\|}\mathbf{k}\right) k_{\|} v_A^2\right]\cdot\delta\mathbf{v}_0 = 0$$

To obtain a nontrivial solution the determinant must vanish, which means

$$\begin{bmatrix} \omega^2 - v_A^2 k_\|^2 - c_{ms}^2 k_\perp^2 & 0 & -c_s^2 k_\| k_\perp \\ 0 & \omega^2 - v_A^2 k_\|^2 & 0 \\ -c_s^2 k_\| k_\perp & 0 & \omega^2 - c_s^2 k_\|^2 \end{bmatrix} \begin{bmatrix} \delta v_{0x} \\ \delta v_{0y} \\ \delta v_{0\|} \end{bmatrix} = 0$$

Here the ***magnetosonic speed*** is given by $c_{ms}^2 = c_s^2 + v_A^2$. The wave vector component perpendicular to the field is oriented along the x-axis, $\mathbf{k} = k_{||}\hat{\mathbf{e}}_z + k_\perp \hat{\mathbf{e}}_x$.

Alfvén Waves

Inspection of the determinant shows that the fluctuation in the *y*-direction decouples from the other two components and has the linear dispersion

$$\omega_A = \pm k_{\|} v_A$$

This ***transverse wave*** travels parallel to the field. It is called ***shear Alfvén wave***. It has no density fluctuation and a constant group velocity, $\mathbf{v}_{gr,A} = \mathbf{v}_A$, which is always oriented along the background field, along which the wave energy is transported.

*The transverse velocity and **magnetic field components are (anti)-correlated** according to:* $\delta v_y/v_A = \pm\, \delta B_y/B_0$, *for **parallel (anti-parallel) wave propagation**. The wave electric field points in the x-direction:* $\delta E_x = \delta B_y/v_A$

Magnetosonic Waves

The remaing four matrix elements couple the fluctuation components, $\delta v_{||}$ and $\delta v_{\perp}$. The corresponding determinant reads:

$$\omega^4 - \omega^2 c_{ms}^2 k^2 + c_s^2 v_A^2 k^2 k_{\|}^2 = 0$$

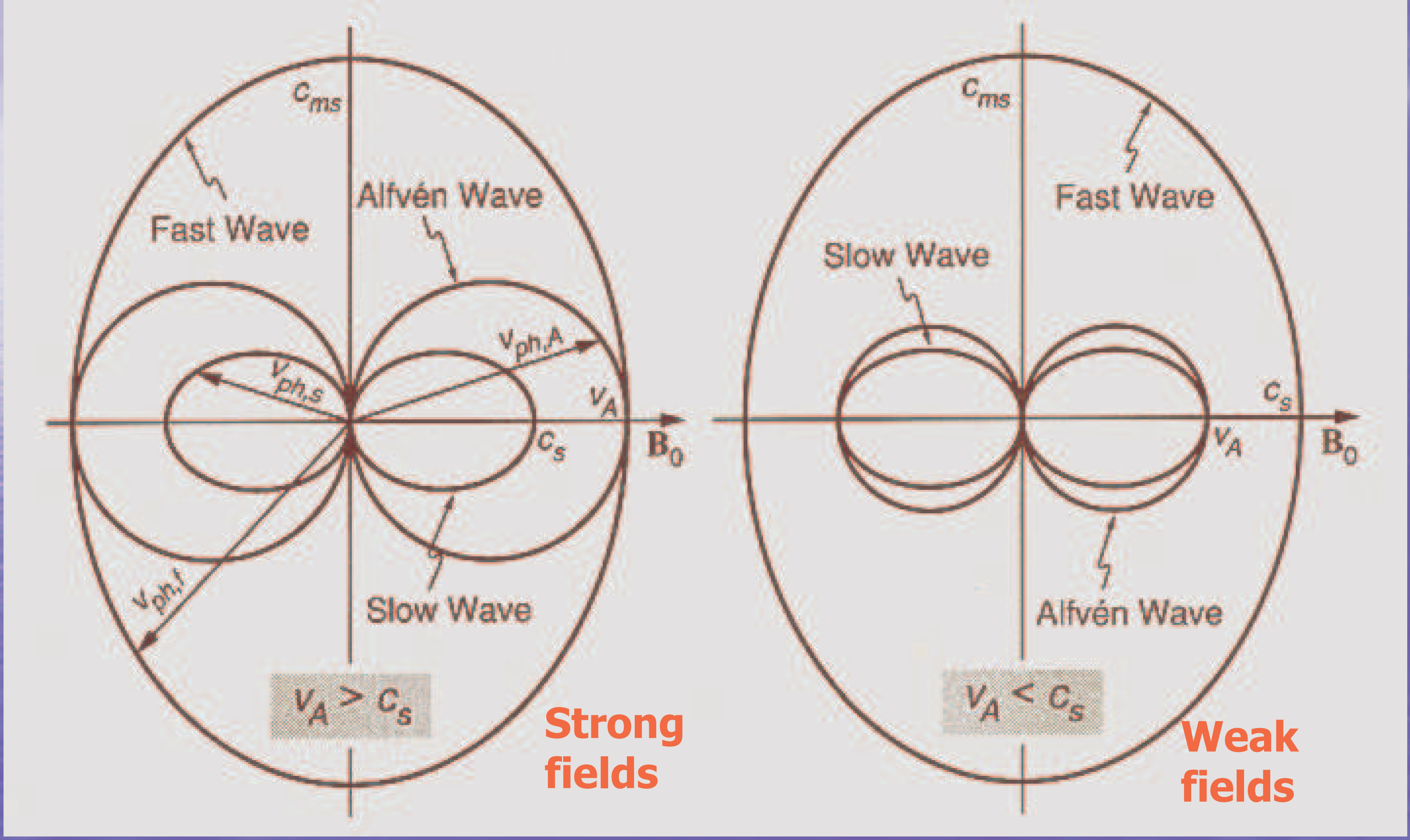
This bi-quadratic equation has the roots:

$$\omega_{ms}^2 = \frac{k^2}{2}\left\{c_{ms}^2 \pm \left[\left(v_A^2 - c_s^2\right)^2 + 4 v_A^2 c_s^2 \frac{k_{\perp}^2}{k^2}\right]^{1/2}\right\}$$

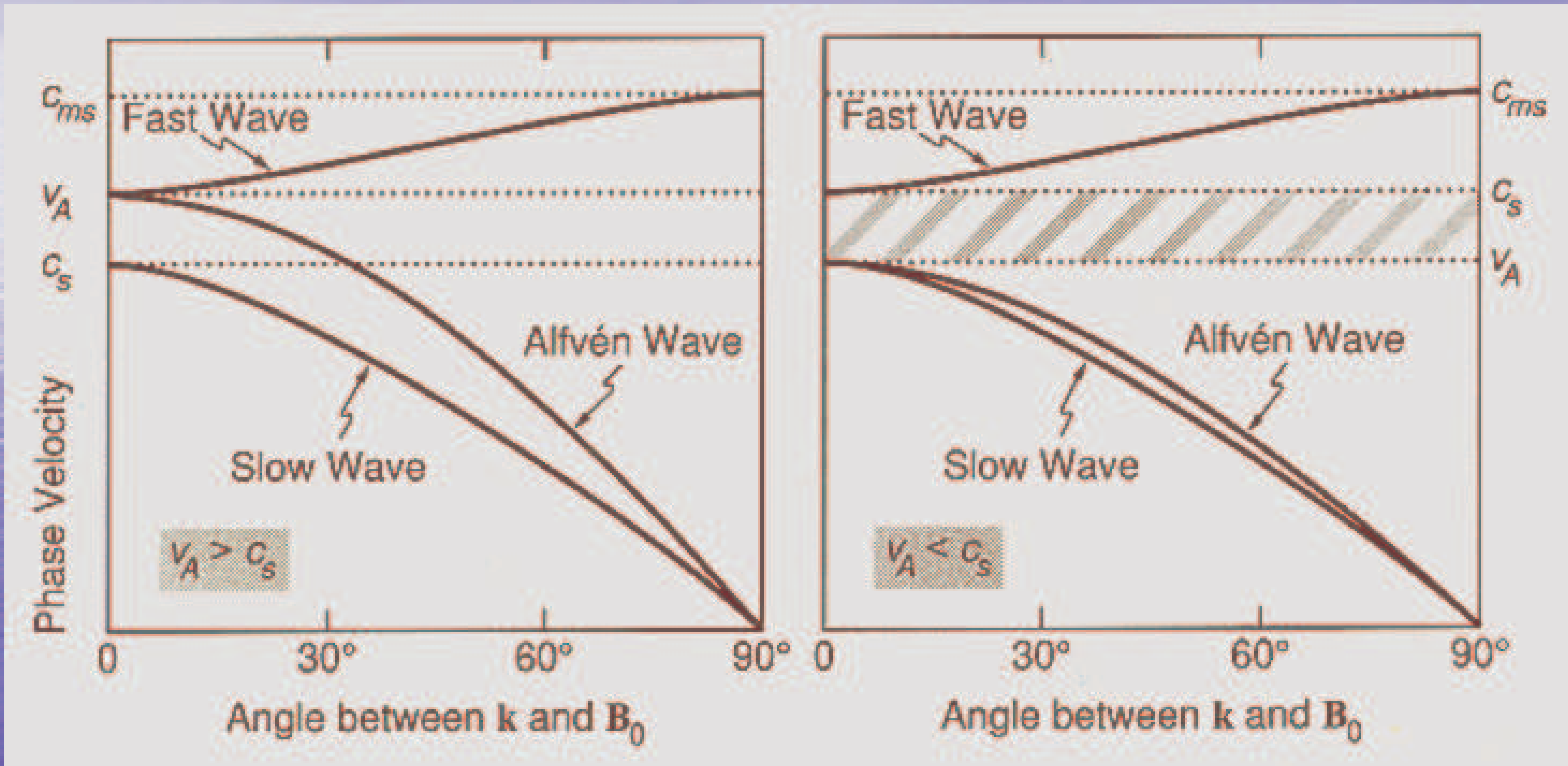
which are the phase velocities of the compressive ***fast and slow magnetosonic waves***. They depend on the propagation angle θ, with $k_{\perp}^2/k^2 = \sin^2\theta$. For $\theta = 90^0$ we have: $\omega = kc_{ms}$, and $\theta = 0^0$:

$$\omega^2 = \frac{1}{2}k^2\left[c_s^2 + v_A^2 \pm \left(c_s^2 - v_A^2\right)\right]$$

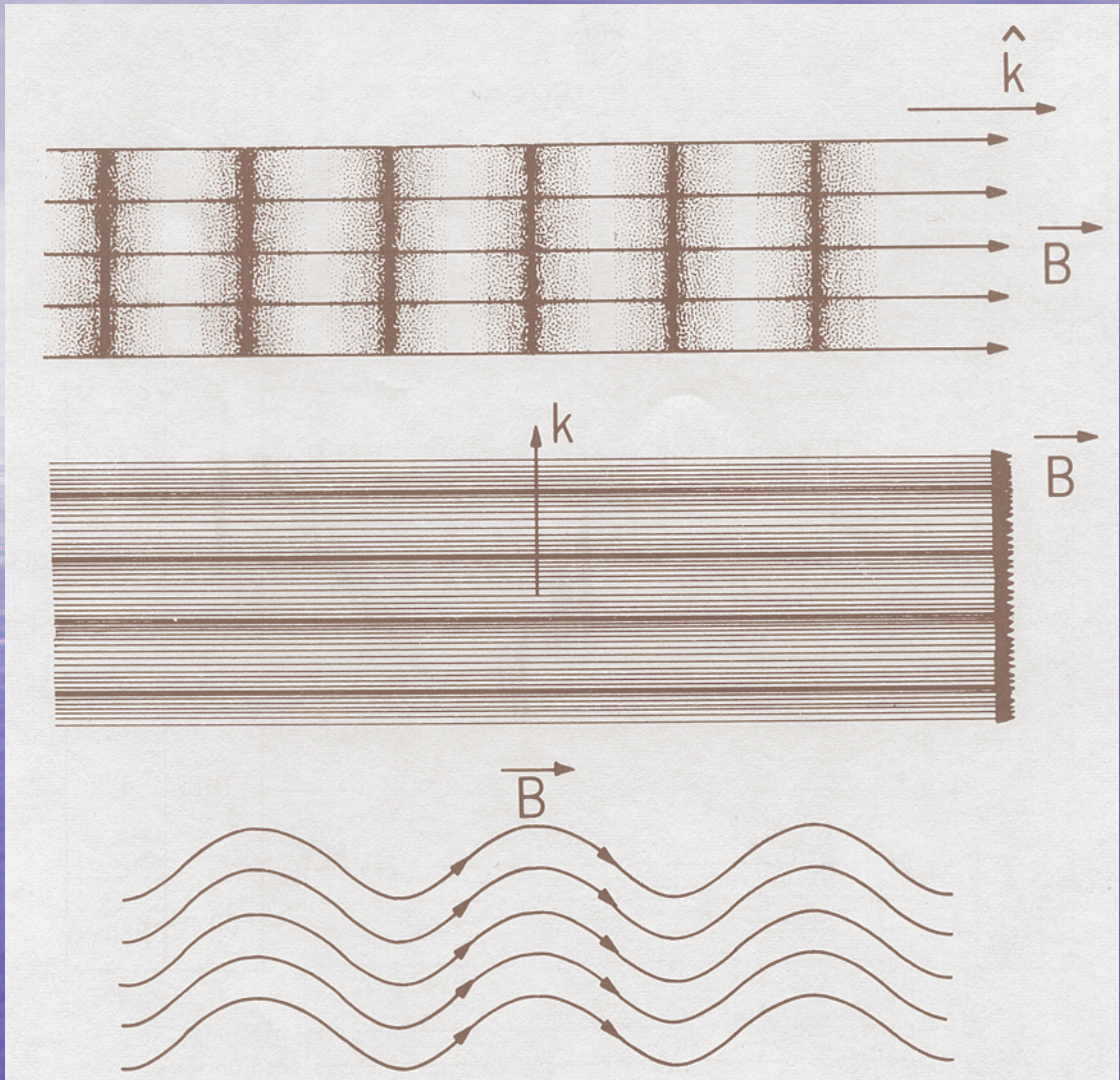
Phase-Velocity Polar Diagram of MHD Waves



Phase Velocity vs Propagation Angle



Magnetohydrodynamic Waves



• Magnetosonic waves

compressible

- parallel slow and fast

- perpendicular fast

$c_{ms} = (c_s^2 + v_A^2)^{1/2}$

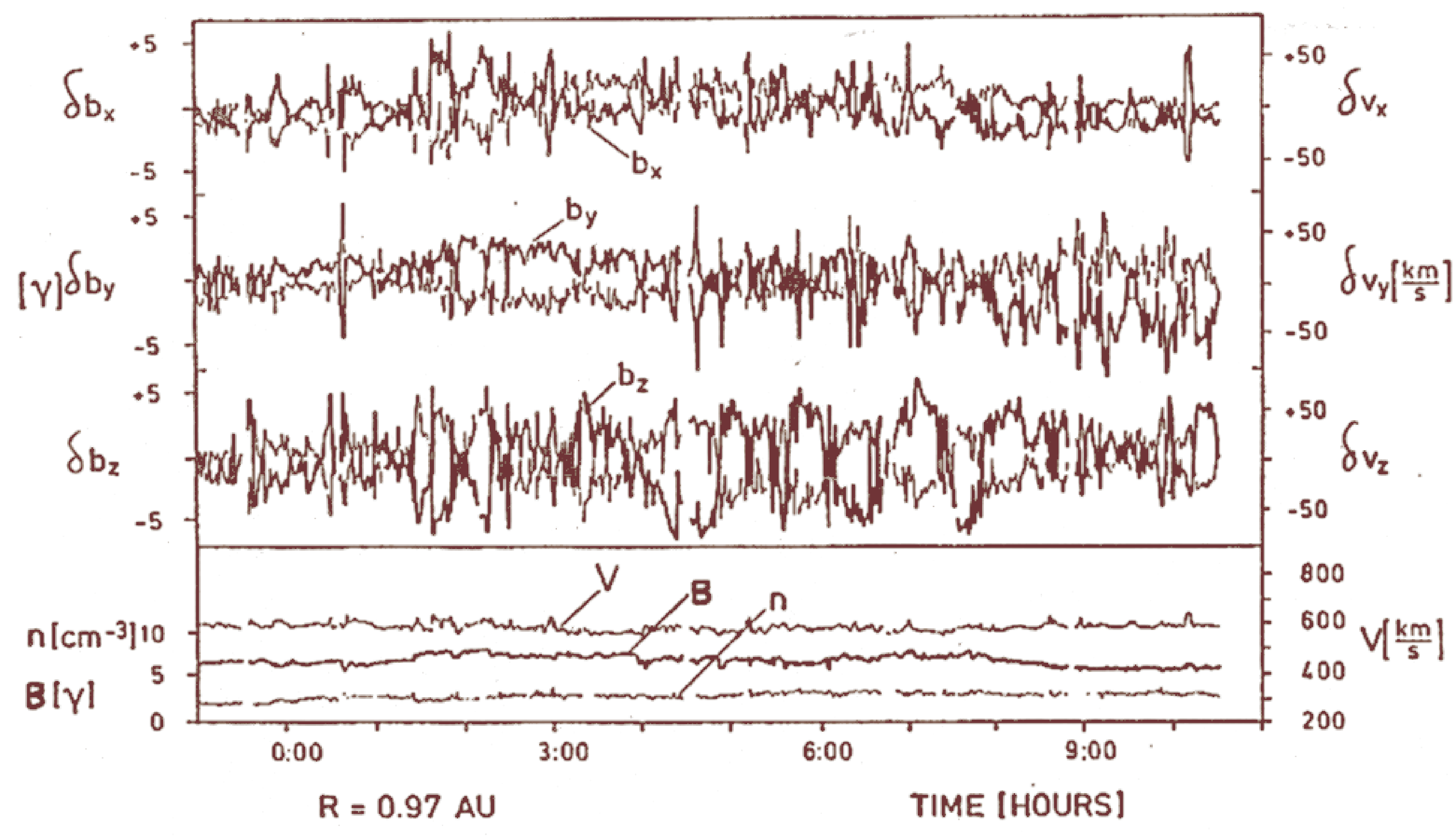
• Alfvén wave

incompressible

parallel and oblique

$v_A = B/(4\pi\rho)^{1/2}$

Alfvén waves in the solar wind

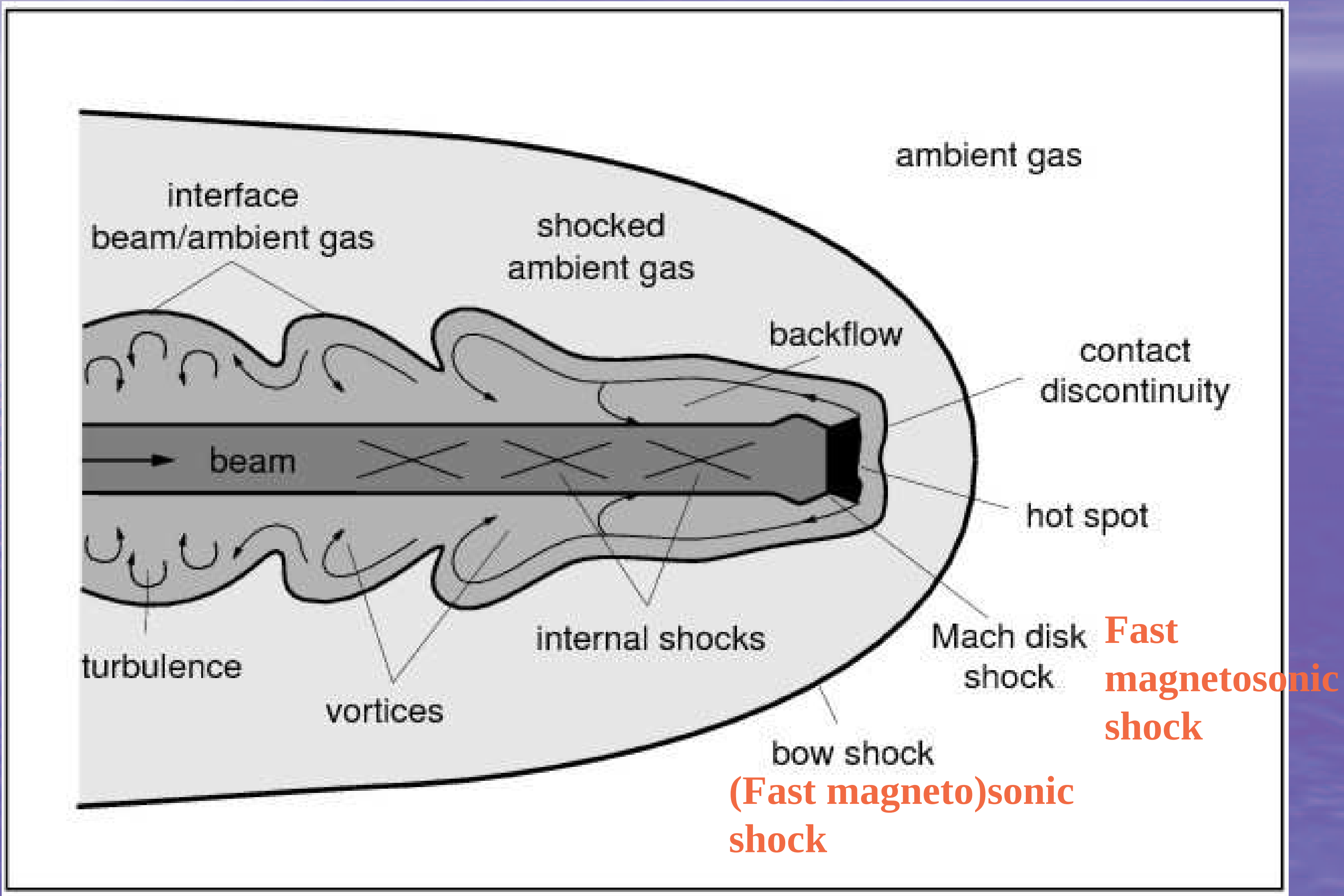


Neubauer et al., 1977

Helios

$\delta v = \pm \delta v_A$

Jet – Anatomy - Shocks



Ad Div(B) Problem

- 4 methods have been introduced to handle the div(**B**) = 0 problem:
- (i) Projection (Brackbill & Barnes 1980)
- (ii) Constrained flux transport (CFT) (Evans & Hawley 1992)
  → staggered mesh (ZEUS, NIRVANA codes)
  → Dai & Woodward (1998),
  Ruy et al. (1998),
  Balsara & Spicer (2000)

- (iii) 8-wave formulation by Powell (1994, 1996)
  → this requires the addition of some source term for div(**B**) and a propagation equation.
  → only conservative schemes can handle jump conditions correctly !
- (iv) Write **B** in terms of vector potential **A**,
  B = rot **A** → complicated equations

Different Coordinate Systems

In this work, we are primarily interested in three coordinate systems: Cartesian coordinates for which

$$(x_1, x_2, x_3) = (x, y, z), \qquad (h_1, h_2, h_3) = (1, 1, 1),$$

cylindrical coordinates for which

$$(x_1, x_2, x_3) = (r, z, \phi), \qquad (h_1, h_2, h_3) = (1, 1, r),$$

and spherical polar coordinates for which

$$(x_1, x_2, x_3) = (r, \theta, \phi), \qquad (h_1, h_2, h_3) = (1, r, r \sin\theta),$$

where h_i are the metric scale factors. If not stated otherwise, x_3 is always the ignorable coordinates for a reduced 2-D problem, The conservative form of the MHD equations are a slight different for different geometries, because the divergence operator $\nabla\cdot$, gradient operator ∇, and curl operation $\nabla\times$ have different forms in different coordinates. Basically for the gradient of a scalar function, we have

$$\nabla f = \left(\frac{1}{h_1}\frac{\partial f}{\partial x_1}, \frac{1}{h_2}\frac{\partial f}{\partial x_2}, \frac{1}{h_3}\frac{\partial f}{\partial x_3}\right);$$

for the divergence of a vector $A = (a_1, a_2, a_3)$, we have

$$\nabla \cdot A = \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}(h_2 h_3 a_1) + \frac{\partial}{\partial x_2}(h_1 h_3 a_2) + \frac{\partial}{\partial x_3}(h_1 h_2 a_3)\right);$$

for the curl of the vector A, we have

$$\begin{aligned} \nabla \times A \quad = \quad & \left\{\frac{1}{h_2 h_3}\left(\frac{\partial}{\partial x_2}(h_3 a_3) - \frac{\partial}{\partial x_3}(h_2 a_2)\right),\right. \\ & \frac{1}{h_1 h_3}\left(\frac{\partial}{\partial x_3}(h_1 a_1) - \frac{\partial}{\partial x_1}(h_3 a_3)\right), \\ & \left.\frac{1}{h_1 h_2}\left(\frac{\partial}{\partial x_1}(h_2 a_2) - \frac{\partial}{\partial x_2}(h_1 a_1)\right)\right\}. \end{aligned}$$

$$\frac{\partial \mathbf{q}}{\partial t} + \frac{1}{h_1 h_2 h_3}\left(\frac{\partial}{\partial x_1}(h_2 h_3 \mathbf{F}) + \frac{\partial}{\partial x_2}(h_1 h_3 \mathbf{G}) + \frac{\partial}{\partial x_3}(h_1 h_2 \mathbf{H})\right) = \mathbf{S}, \tag{5}$$

where

$$\mathbf{q} = (\rho, \rho v_1, \rho v_2, \rho v_3, B_1, B_2, B_3, E)^t,$$

State Vector

and the flux functions are

$$\mathbf{F} = \begin{pmatrix} \rho v_1 \\ \rho v_1^2 - B_1^2 + p^* \\ \rho v_1 v_2 - B_1 B_2 \\ \rho v_1 v_3 - B_1 B_3 \\ 0 \\ \Omega_3 \\ -\Omega_2 \\ (E + p^*) v_1 - B_1 (\mathbf{B} \cdot \mathbf{v}) \end{pmatrix}, \quad \mathbf{G} = \begin{pmatrix} \rho v_2 \\ \rho v_2 v_1 - B_2 B_1 \\ \rho v_2^2 - B_2^2 + p^* \\ \rho v_2 v_3 - B_2 B_3 \\ -\Omega_3 \\ 0 \\ \Omega_1 \\ (E + p^*) v_2 - B_2 (\mathbf{B} \cdot \mathbf{v}) \end{pmatrix},$$

$$\mathbf{H} = \begin{pmatrix} \rho v_3 \\ \rho v_3 v_1 - B_3 B_1 \\ \rho v_3 v_2 - B_3 B_2 \\ \rho v_3^2 - B_3^2 + p^* \\ \Omega_2 \\ -\Omega_1 \\ 0 \\ (E + p^*) v_3 - B_3 (\mathbf{B} \cdot \mathbf{v}) \end{pmatrix},$$

and the source terms are

$$\mathbf{S} = \begin{pmatrix} 0 \\ \frac{B_1B_2-\rho v_1 v_2}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{B_1B_3-\rho v_1 v_3}{h_1h_3}\frac{\partial h_1}{\partial x_3} + \frac{\rho v_2^2-B_2^2}{h_1h_2}\frac{\partial h_2}{\partial x_1} + \frac{\rho v_3^2-B_3^2}{h_1h_3}\frac{\partial h_3}{\partial x_1} + \frac{p^*}{h_1h_2h_3}\frac{\partial(h_2h_3)}{\partial x_1} \\ \frac{B_2B_1-\rho v_2 v_1}{h_1h_2}\frac{\partial h_2}{\partial x_1} + \frac{B_2B_3-\rho v_2 v_3}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \frac{\rho v_1^2-B_1^2}{h_1h_2}\frac{\partial h_1}{\partial x_2} + \frac{\rho v_3^2-B_3^2}{h_2h_3}\frac{\partial h_3}{\partial x_2} + \frac{p^*}{h_1h_2h_3}\frac{\partial(h_1h_3)}{\partial x_2} \\ \frac{B_3B_2-\rho v_3 v_2}{h_3h_2}\frac{\partial h_3}{\partial x_2} + \frac{B_1B_3-\rho v_1 v_3}{h_1h_3}\frac{\partial h_3}{\partial x_1} + \frac{\rho v_1^2-B_1^2}{h_1h_3}\frac{\partial h_1}{\partial x_3} + \frac{\rho v_2^2-B_2^2}{h_2h_3}\frac{\partial h_2}{\partial x_3} + \frac{p^*}{h_1h_2h_3}\frac{\partial(h_1h_2)}{\partial x_3} \\ \frac{\Omega_2}{h_1h_3}\frac{\partial h_1}{\partial x_3} - \frac{\Omega_3}{h_1h_2}\frac{\partial h_1}{\partial x_2} \\ \frac{\Omega_3}{h_1h_2}\frac{\partial h_2}{\partial x_1} - \frac{\Omega_1}{h_3h_2}\frac{\partial h_2}{\partial x_3} \\ \frac{\Omega_1}{h_2h_3}\frac{\partial h_3}{\partial x_2} - \frac{\Omega_2}{h_1h_3}\frac{\partial h_3}{\partial x_1} \\ 0 \end{pmatrix}$$

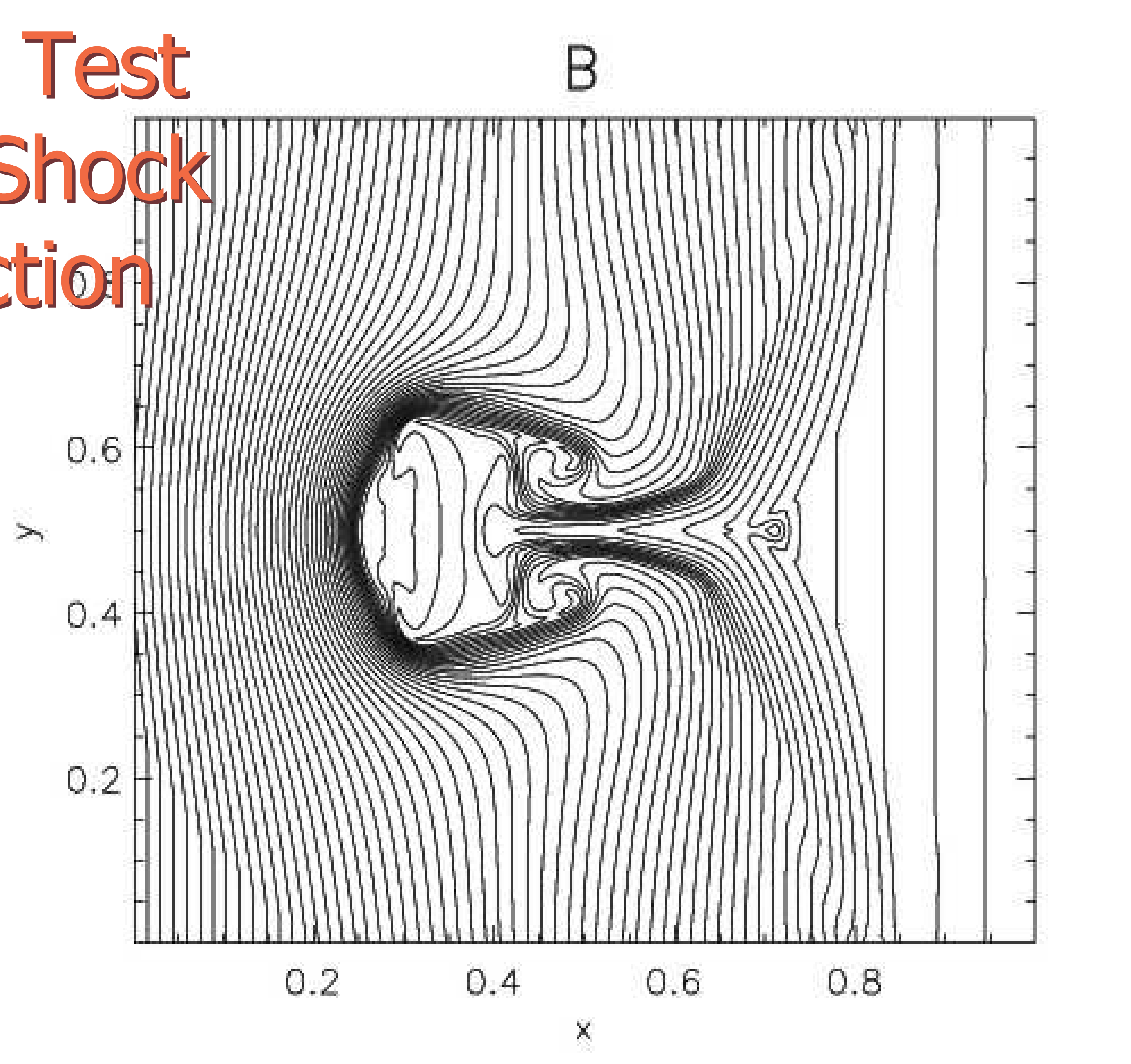
where

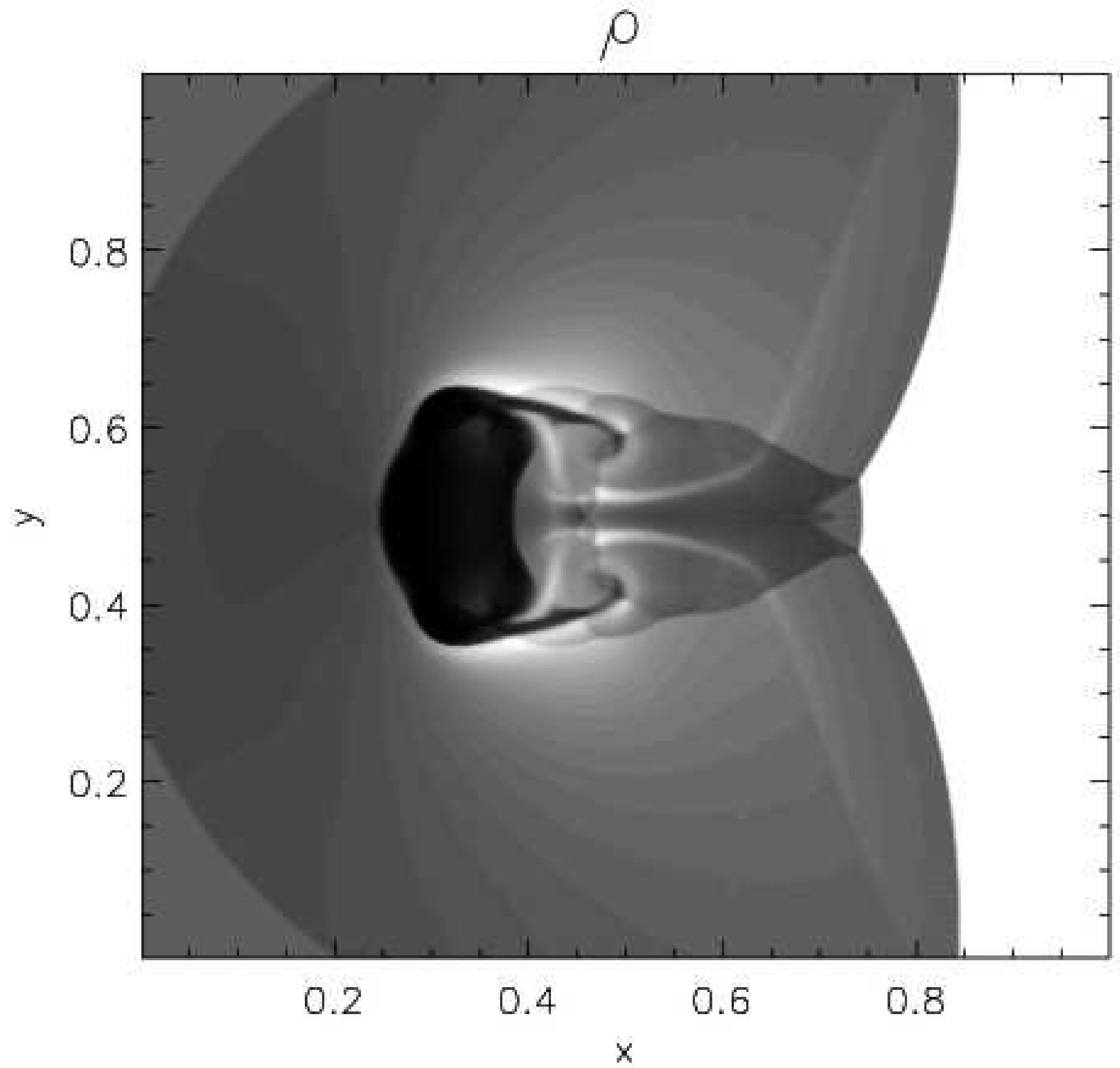
$$p^* = p + \frac{1}{2}\mathbf{B}\cdot\mathbf{B}$$

is the total pressure,

$$\Omega_1 = v_2B_3 - v_3B_2, \quad \Omega_2 = v_3B_1 - v_1B_3, \quad \Omega_3 = v_1B_2 - v_2B_1$$

Crucial Test Cloud Shock Interaction




ρ
0.8
0.6
y
0.4
0.2
0.2
0.4
0.6
0.8
x

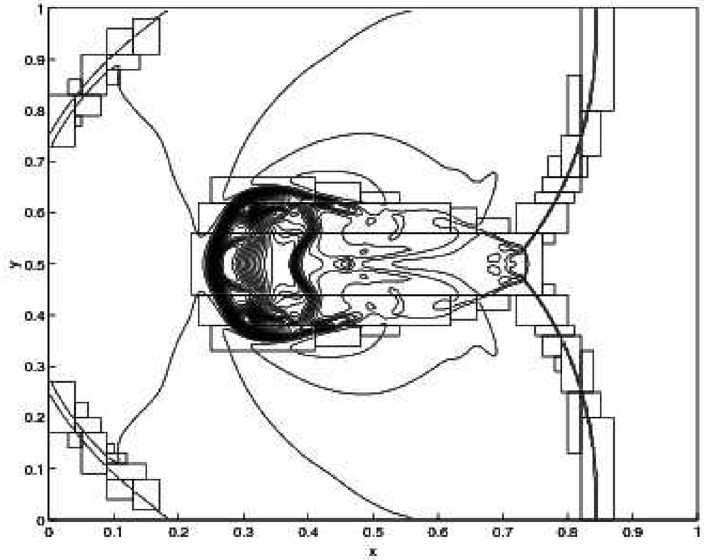


Fig. 7.7-a.— Density contour plot for Roe's method with two-level refinement

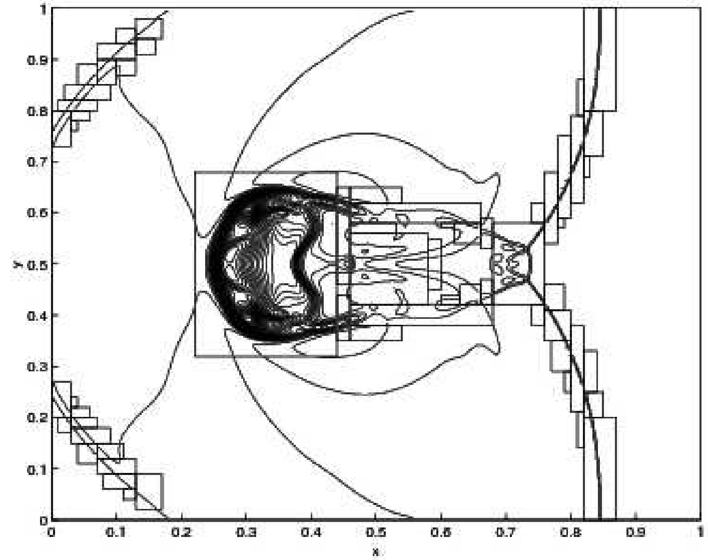


Fig. 7.7-b.— Density contour plot for HLLC solver with two-level refinement.

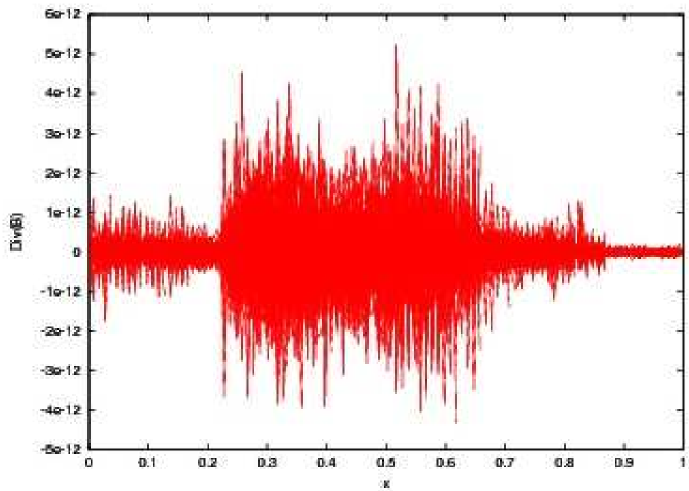


Fig. 7.7-c.— $\nabla \cdot B$ plot for Roe's method with two-level refinement.

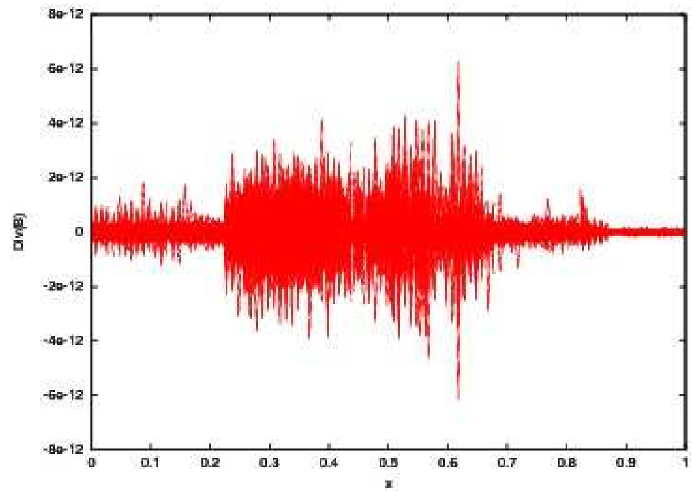


Fig. 7.7-d.— $\nabla \cdot B$ plot for HLLC solver with two-level refinement.

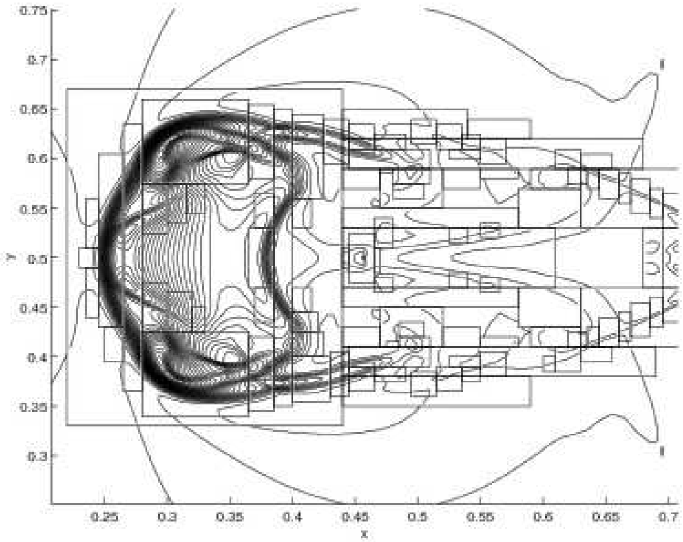


Fig. 7.8-a.— Density contour plot for Roe's method with three-level refinement.

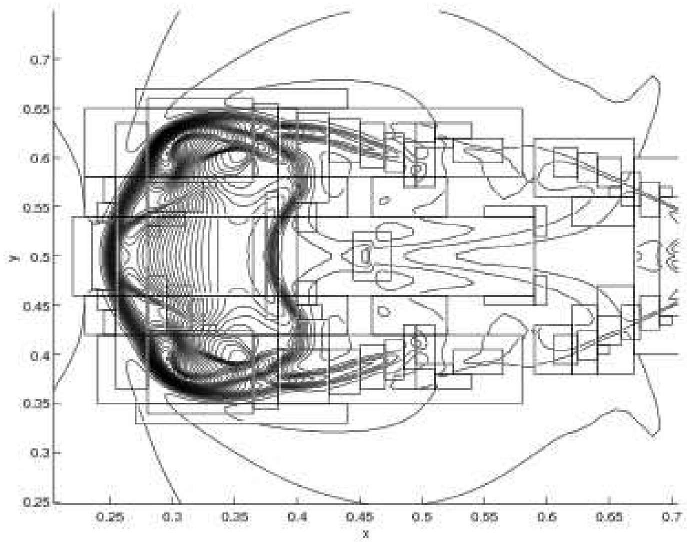


Fig. 7.8-b.— Density contour plot for HLLC solver with three-level refinement.

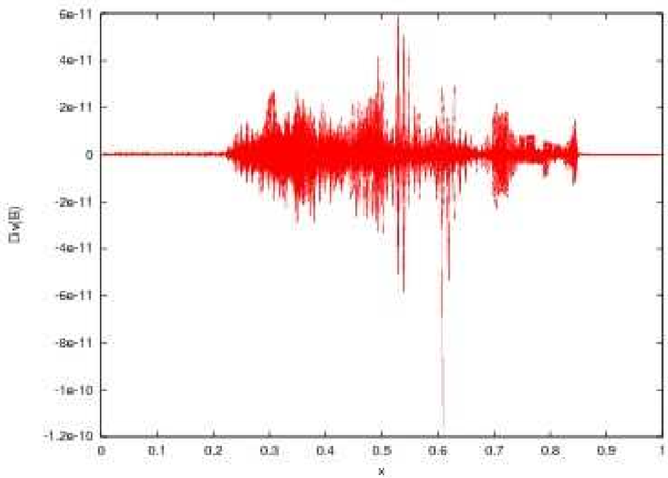


Fig. 7.8-c.— $\nabla \cdot B$ plot for Roe's method with three-level refinement.

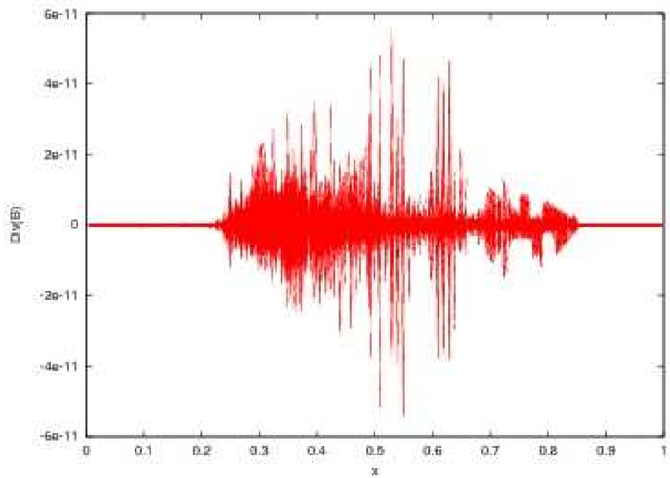
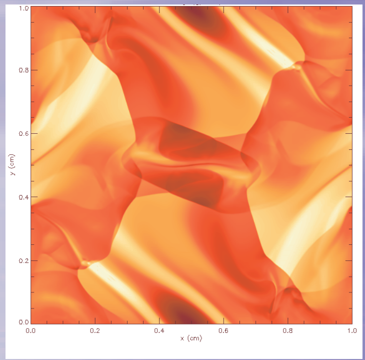


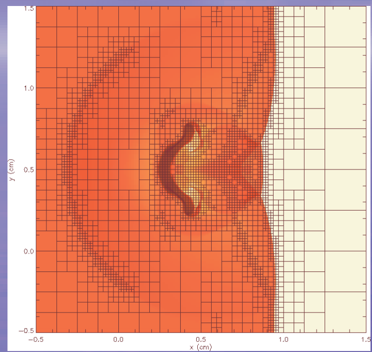
Fig. 7.8-d.— $\nabla \cdot B$ plot for HLLC solver with three-level refinement.

Tests considered by Flash

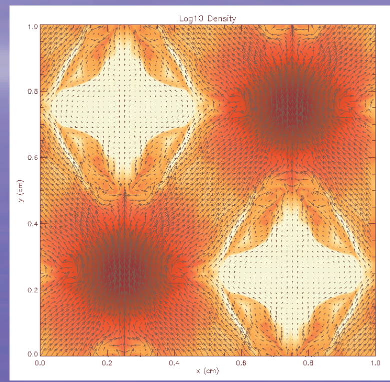
Orszag-Tang



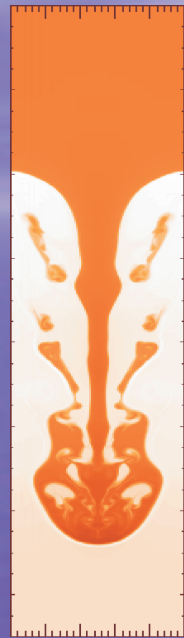
Shock-Cloud Interaction



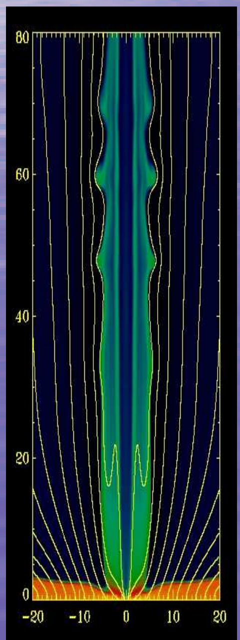
Self-Gravitating Plasma



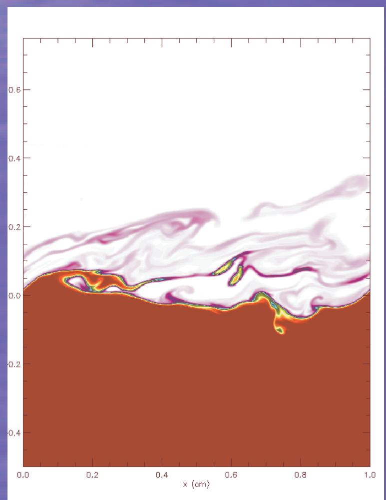
Magnetic RT



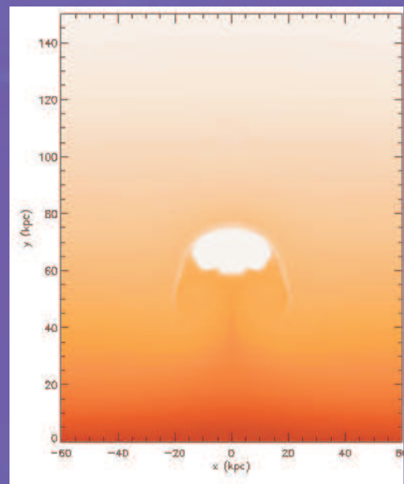
Jet Launching



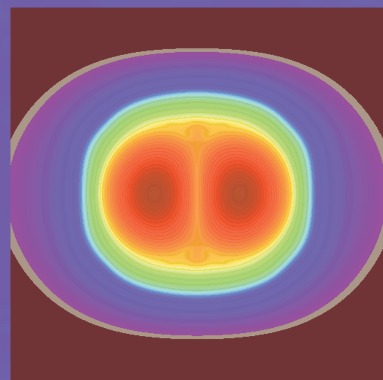
Surface Gravity Wave



Rising bubble



Magnetic reconnection

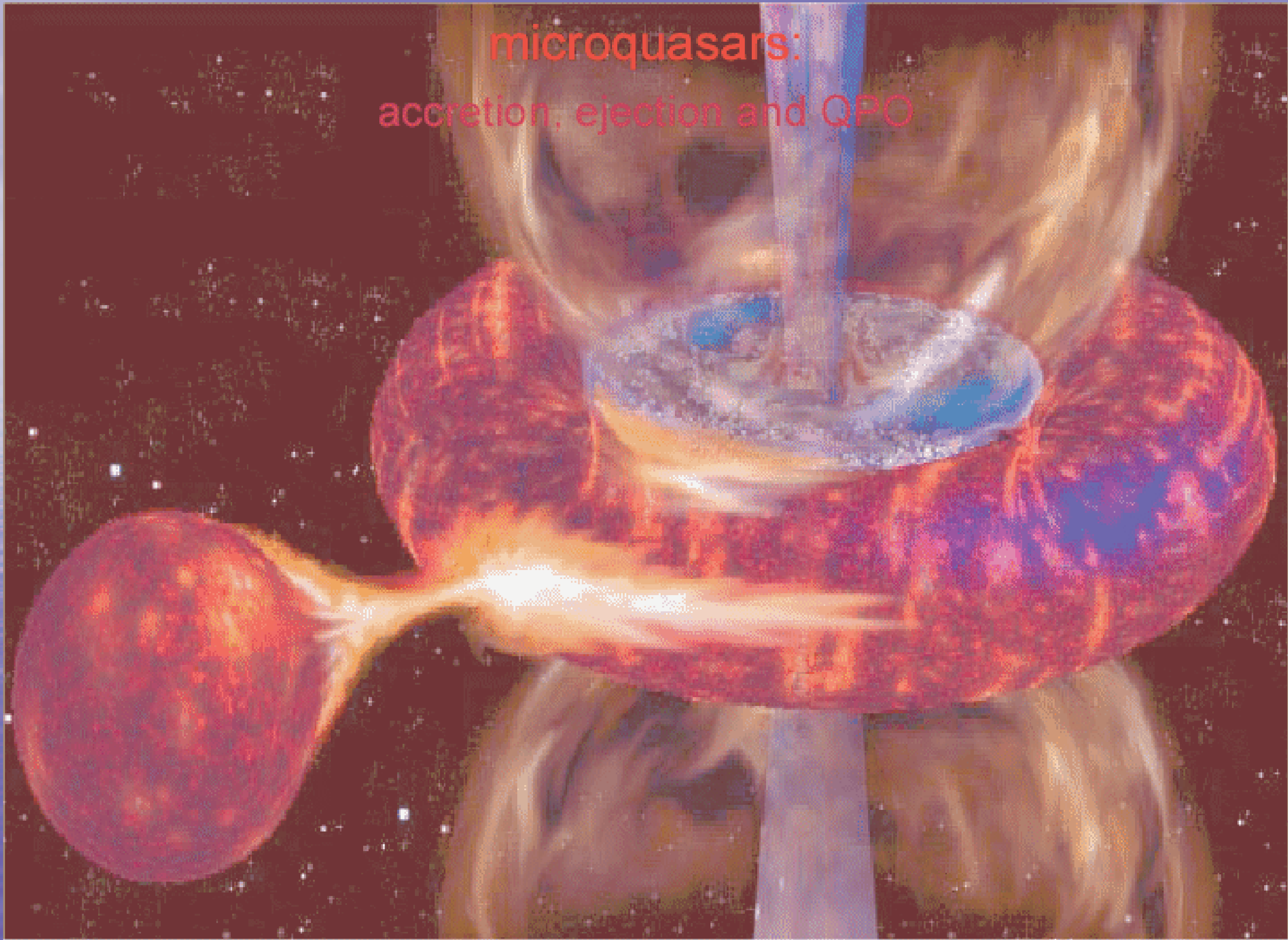


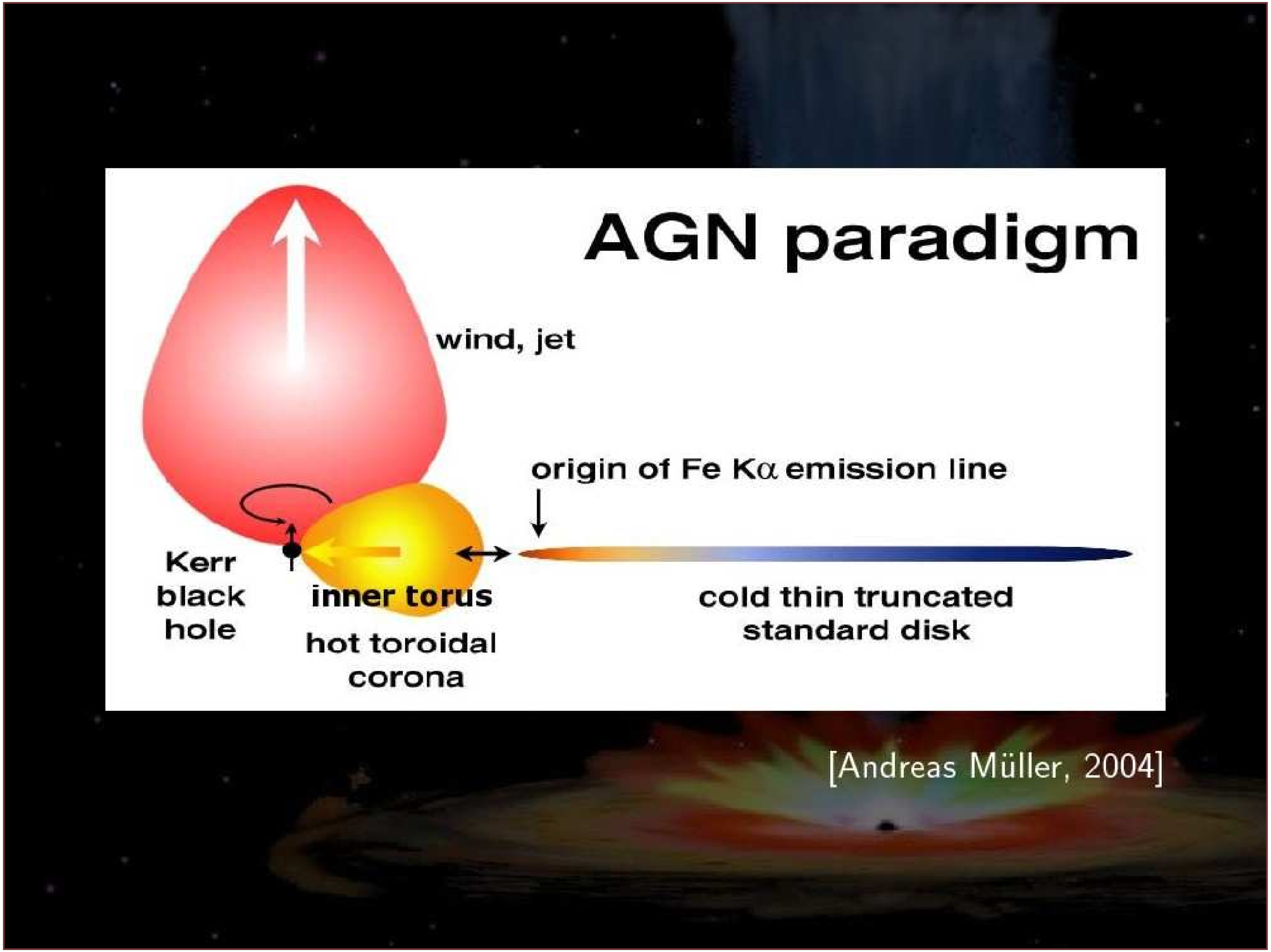
Application I: Disk Simulations → Magnetorotational Instability

- Weakly magnetized disks are unstable
- Weak field limit of MHD: $\beta \sim 100 - 1000$
- Balbus-Hawley instability (Balbus & Hawley 1991; 1998; 2003)
- Initially, only 2D shear box simulations
- Since 2000, global simulations, with initial condition given by some torus distribution
- Challenge: Exact angular momentum conservation !

microquasars:

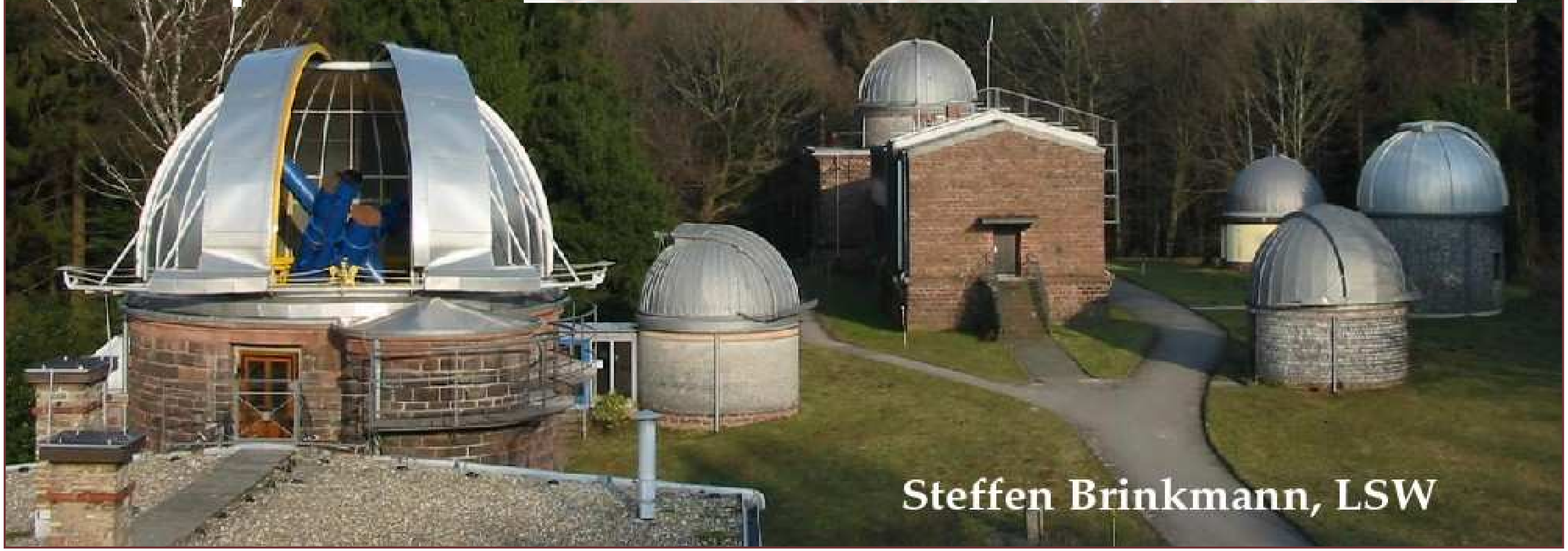
accretion, ejection and QPO




AGN paradigm
wind, jet
origin of Fe Kα emission line
Kerr
black
hole
inner torus
hot toroidal
corona
cold thin truncated
standard disk
[Andreas Müller, 2004]

Simulations of Radiative MHD in Turbulent Accretion discs

NEC SX-8 HLRS
9 GFlops Peak
8 CPUs per Node
Vector CPU
72 Nodes
64 GB per Node



Steffen Brinkmann, LSW

Astrophysical Context

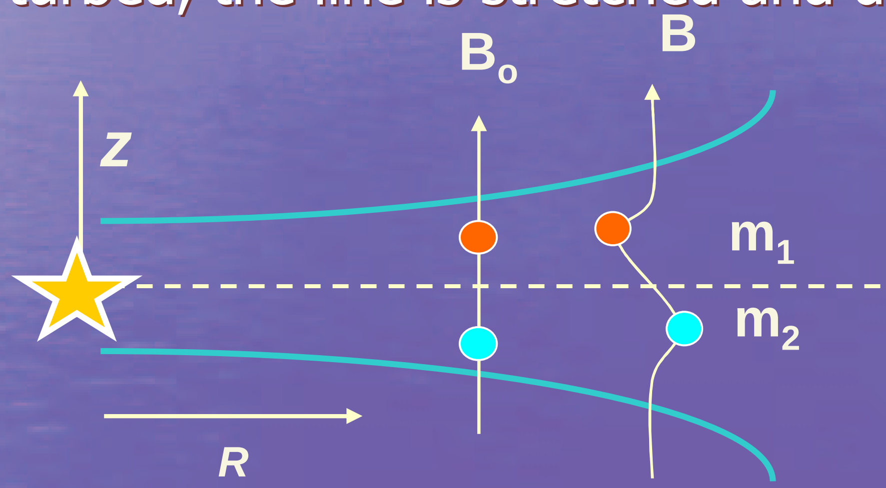
- What drives the turbulence in disks ?
- General consensus : Accretion disks ARE turbulent, but because of the pervasive magnetic fields, not hydrodynamics
- The central idea is that weak magnetic fields cause a linear instability which leads directly to disk turbulence
- Magnetorotational Instability (MRI) likely to be the source of this turbulence and orbital angular momentum transport

Accretion Questions

- What disk instabilities are present?
- What disk structures arise naturally?
- What are the properties of disk turbulence?
- Is there a dynamo?
- How are winds and/or jets produced?
- Origin of QPOs and Fe Ka line
- What are the properties of the inner disk?
- How does black hole spin affect accretion?
- How does accretion affect the black hole?

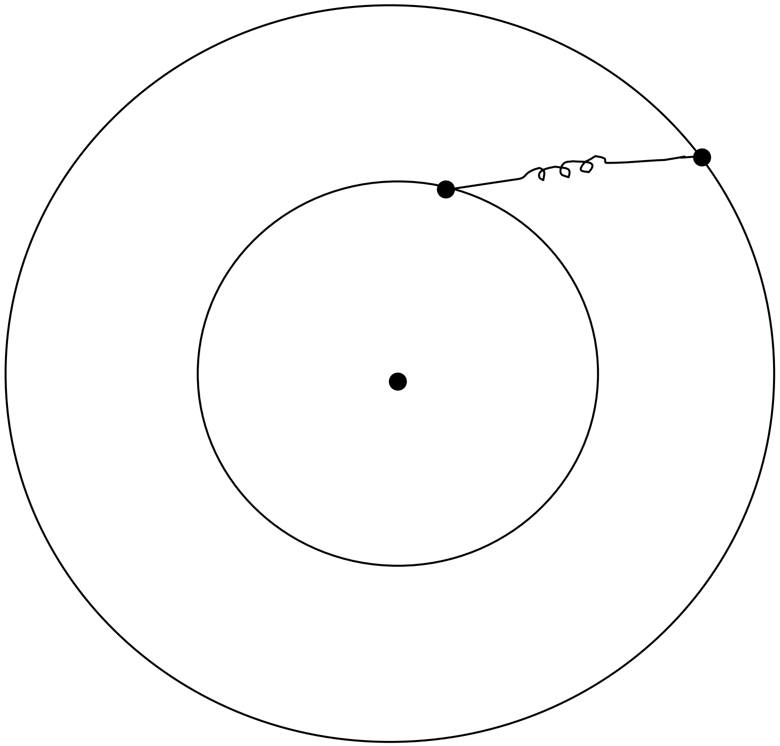
Balbus-Hawley's Solution

- Two fluid elements, in the same orbit, are joined by a field line (B_o). The tension in the line is negligible.
- If they are perturbed, the line is stretched and develops tension.



- The tension acts to reduce the angular momentum of m_1 and increase that of m_2. This further increases the tension and the process "runs away".

MRI (Balbus and Hawley, 1991)



- Keplerian radial profile : $\Omega(r) \propto r^{-\frac{3}{2}}$
- Fluid elements coupled by a weak spring

  with spring constant = $(\mathbf{k} \cdot \mathbf{V}_A)^2$

  when weak magnetic field is applied
- Spring exerts a tension force on both

  elements, transferring angular momentum

  from the inner element to the outer
- Instability arises as element separation increases, causing tension to increase,

  causing further element separation, etc. (runaway process)

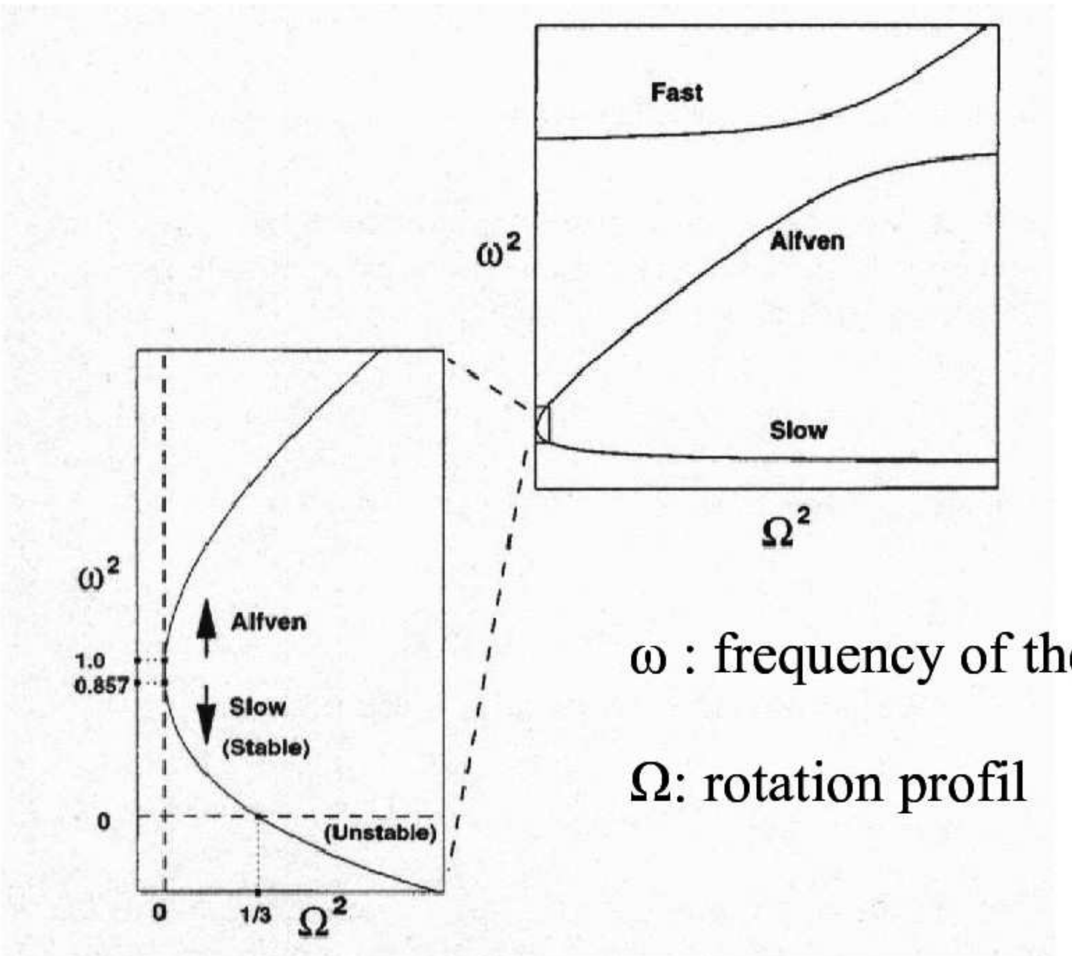
Dispersion relation

After some calculation one finds

$$[\omega^2 - (\mathbf{k}\cdot\mathbf{u_A})^2][\omega^4 - k^2\omega^2(a^2 + u_A^2) + (\mathbf{k}\cdot\mathbf{u_A})^2 k^2 a^2]$$
$$- \left[\kappa^2\omega^4 - \omega^2 \left(k^2\kappa^2(a^2 + u_{A\phi}^2) + (\mathbf{k}\cdot\mathbf{u_A})^2 \frac{d\Omega^2}{d \ln R} \right) \right]$$
$$- k^2 a^2 (\mathbf{k}\cdot\mathbf{u_A})^2 \frac{d\Omega^2}{d \ln R} = 0$$

from which the three magnetosonic waves can be deduced.

Magnetosonic modes



ω : frequency of the waves

Ω: rotation profil

Maximum growth rate

From the analysis of the dispersion relation one finds

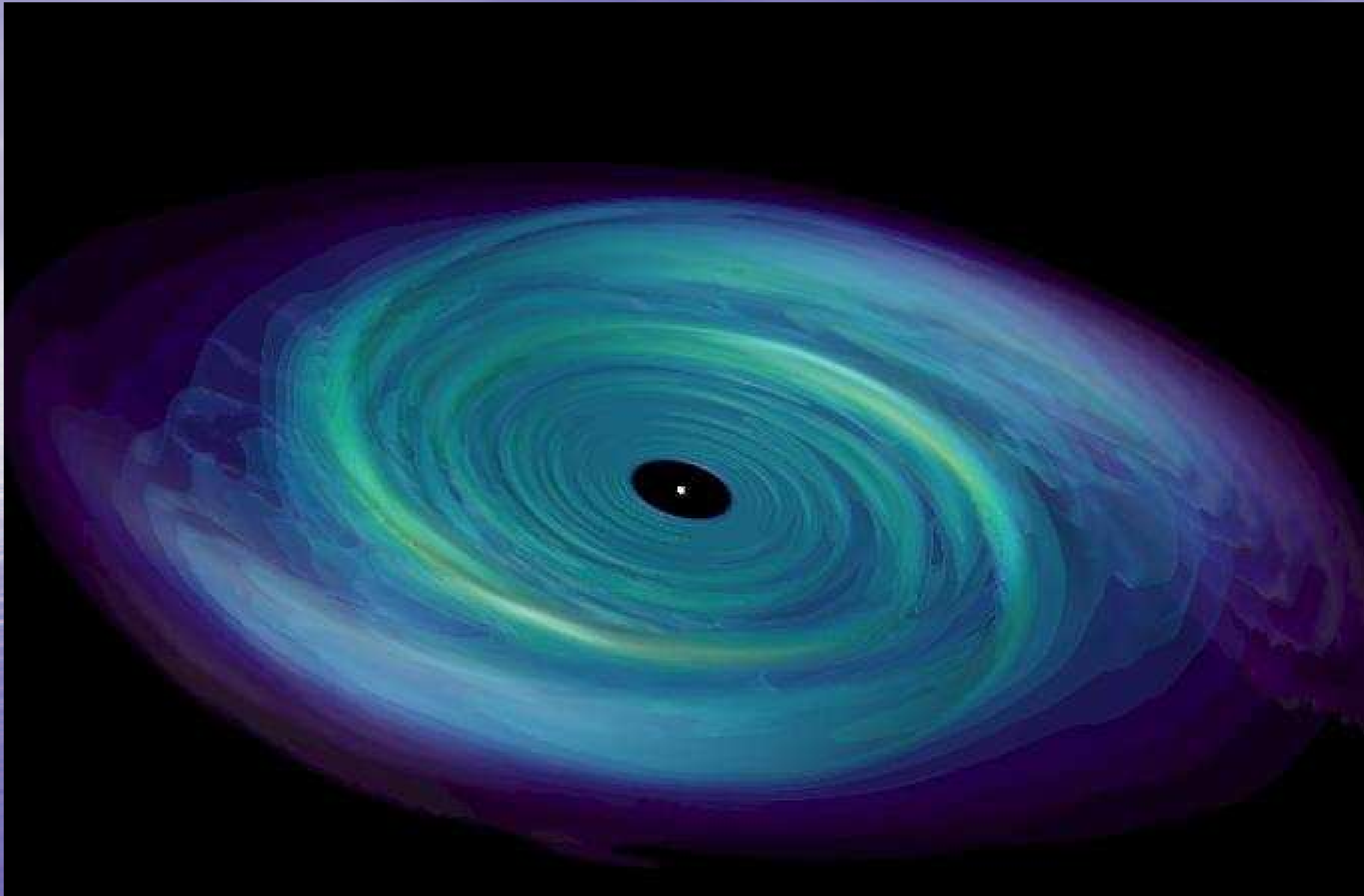
$$|\omega_{max}| = \frac{1}{2}\left|\frac{d\,\Omega}{dln\,R}\right|$$

$$(\boldsymbol{k}\cdot\boldsymbol{u}_A)^2_{max} = -\left(\frac{1}{4}+\frac{\kappa^2}{16}\Omega^2\right)\frac{d\,\Omega}{dlnR}$$

which means for the cases of keplerian disks

$$|\omega_{max}| = \frac{3}{4}\Omega \qquad (\boldsymbol{k}\cdot\boldsymbol{u}_A)_{max} = \frac{\sqrt{15}}{4}\Omega$$

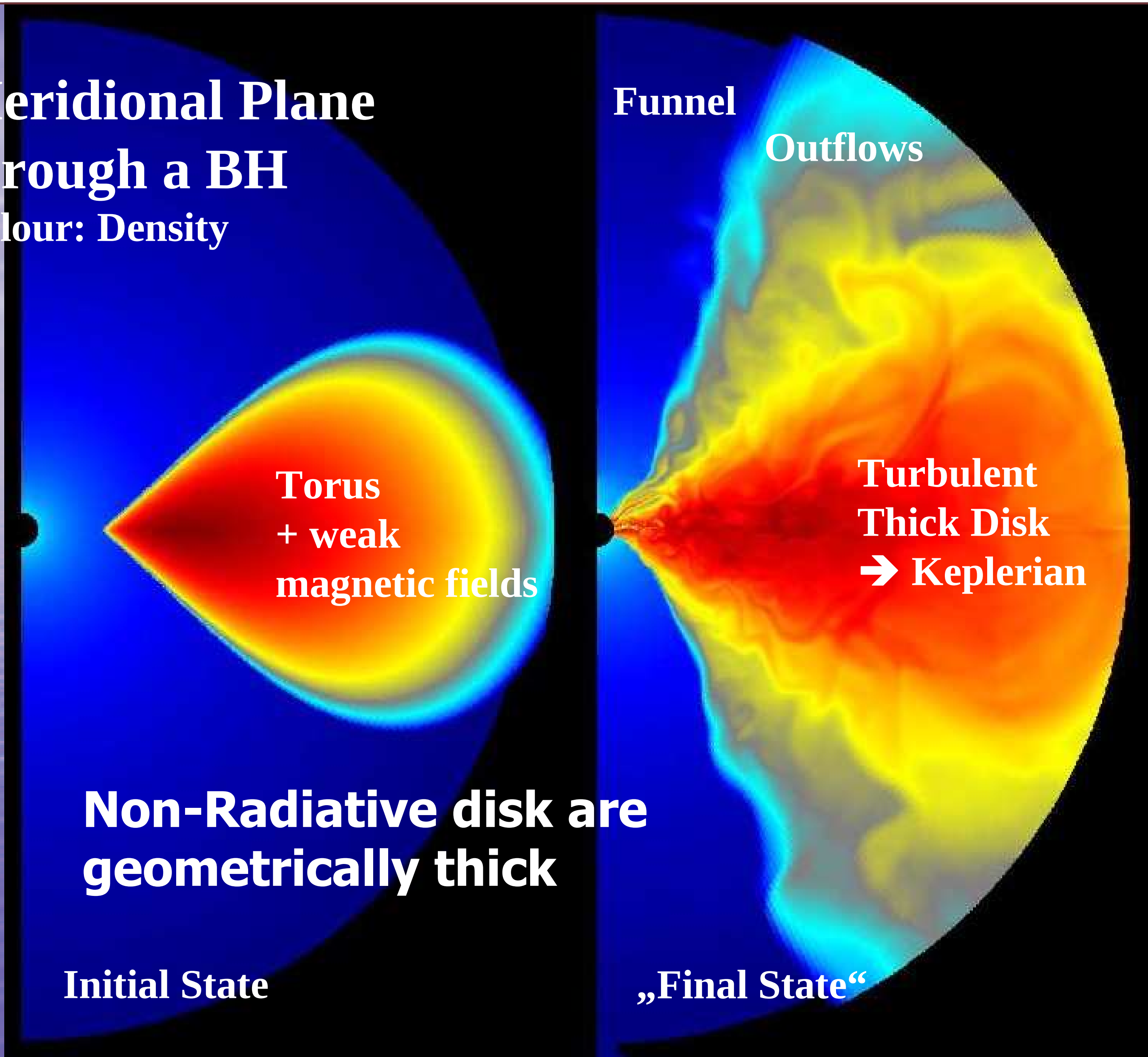
“High Resolution 3D Hydrodynamic Simulations of Accretion Disks in Close Binaries” -- M.P.Owen



Radiative disks are geometrically thin

Meridional Plane through a BH

Colour: Density



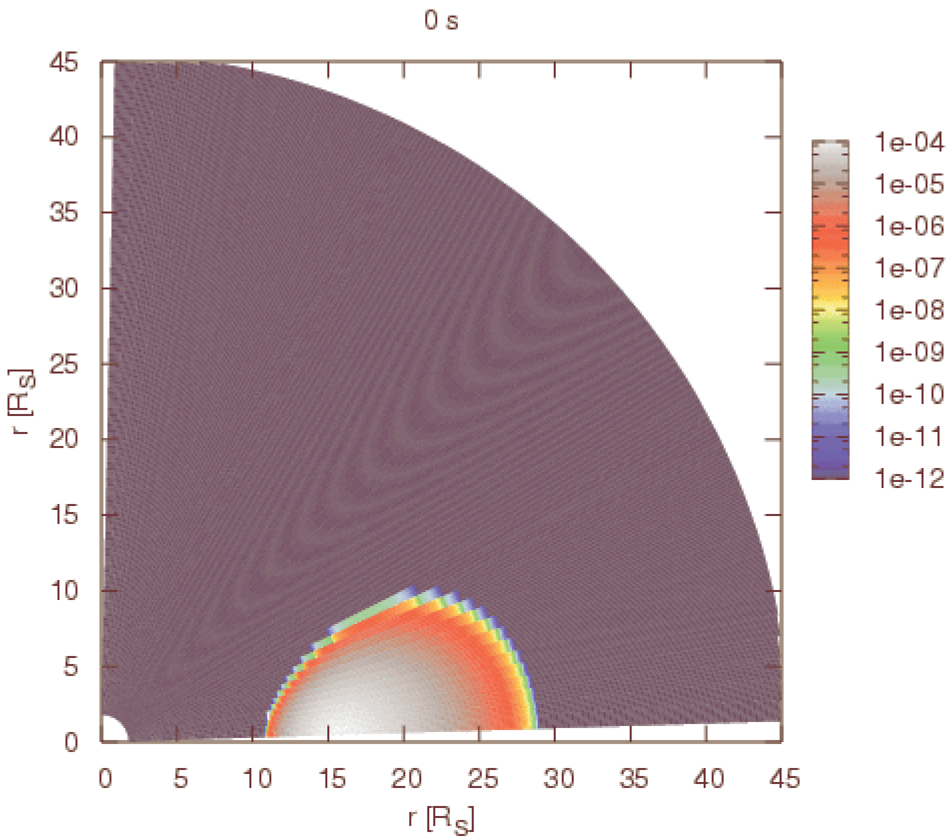
Non-Radiative disk are geometrically thick

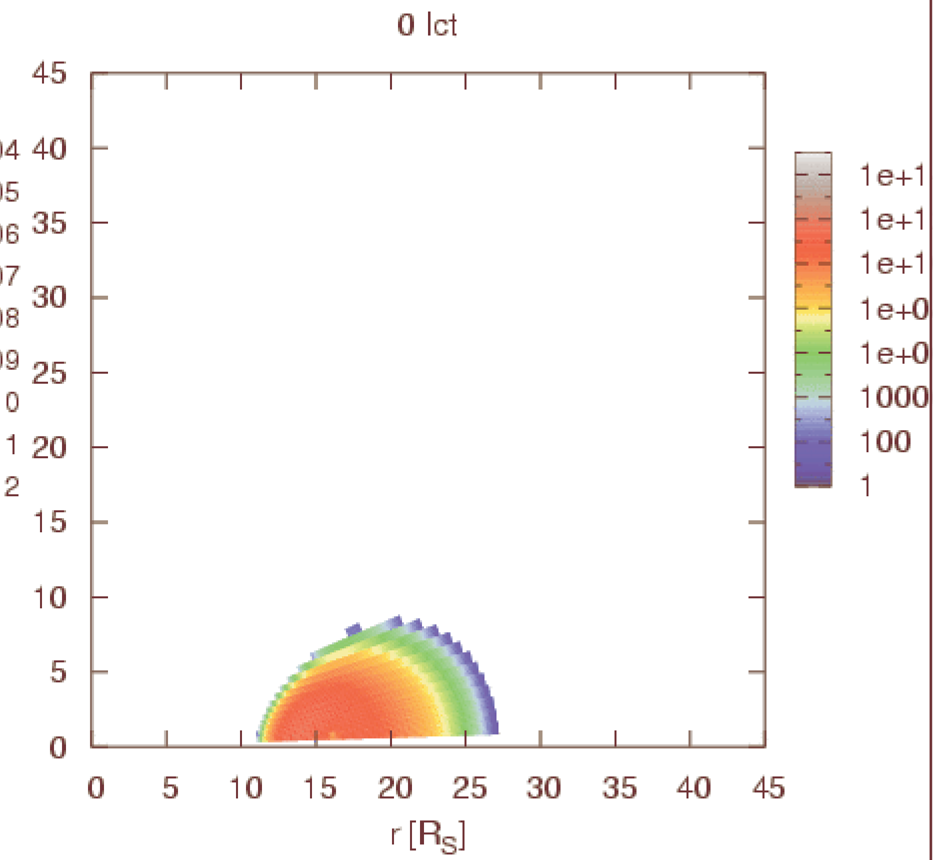
Gammie et al. 2004

No cooling

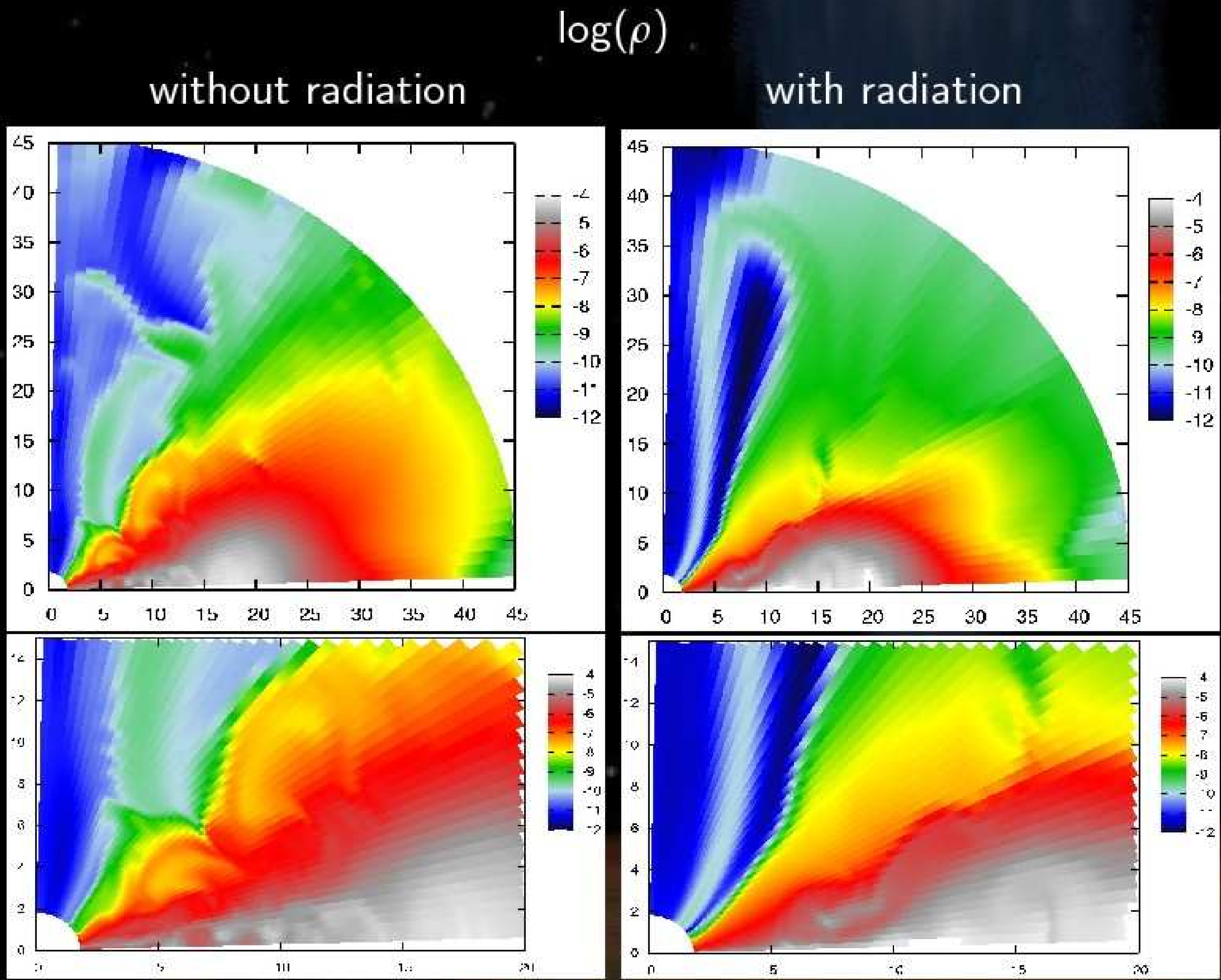
Cyclotron cooling

Log ρ

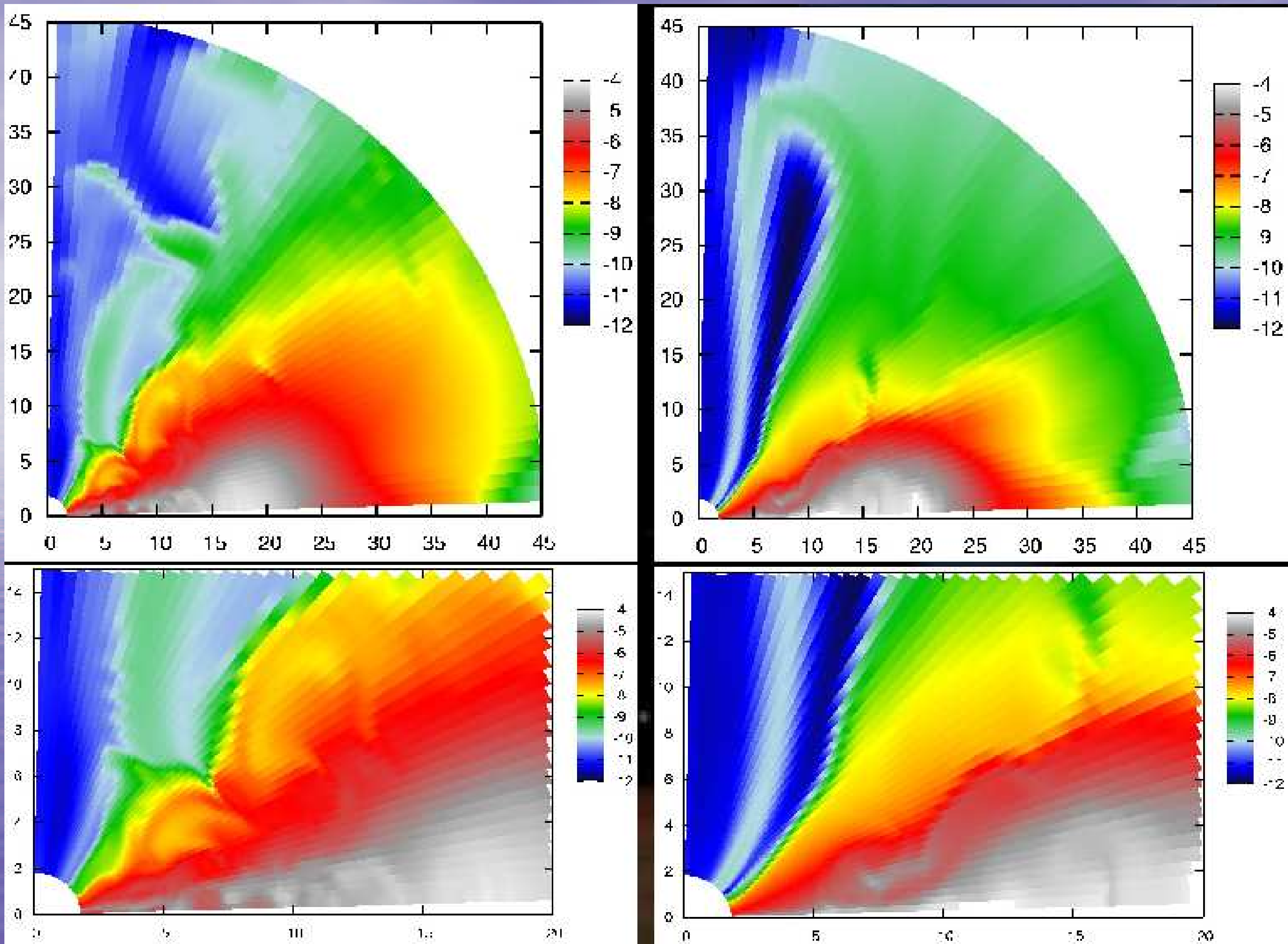




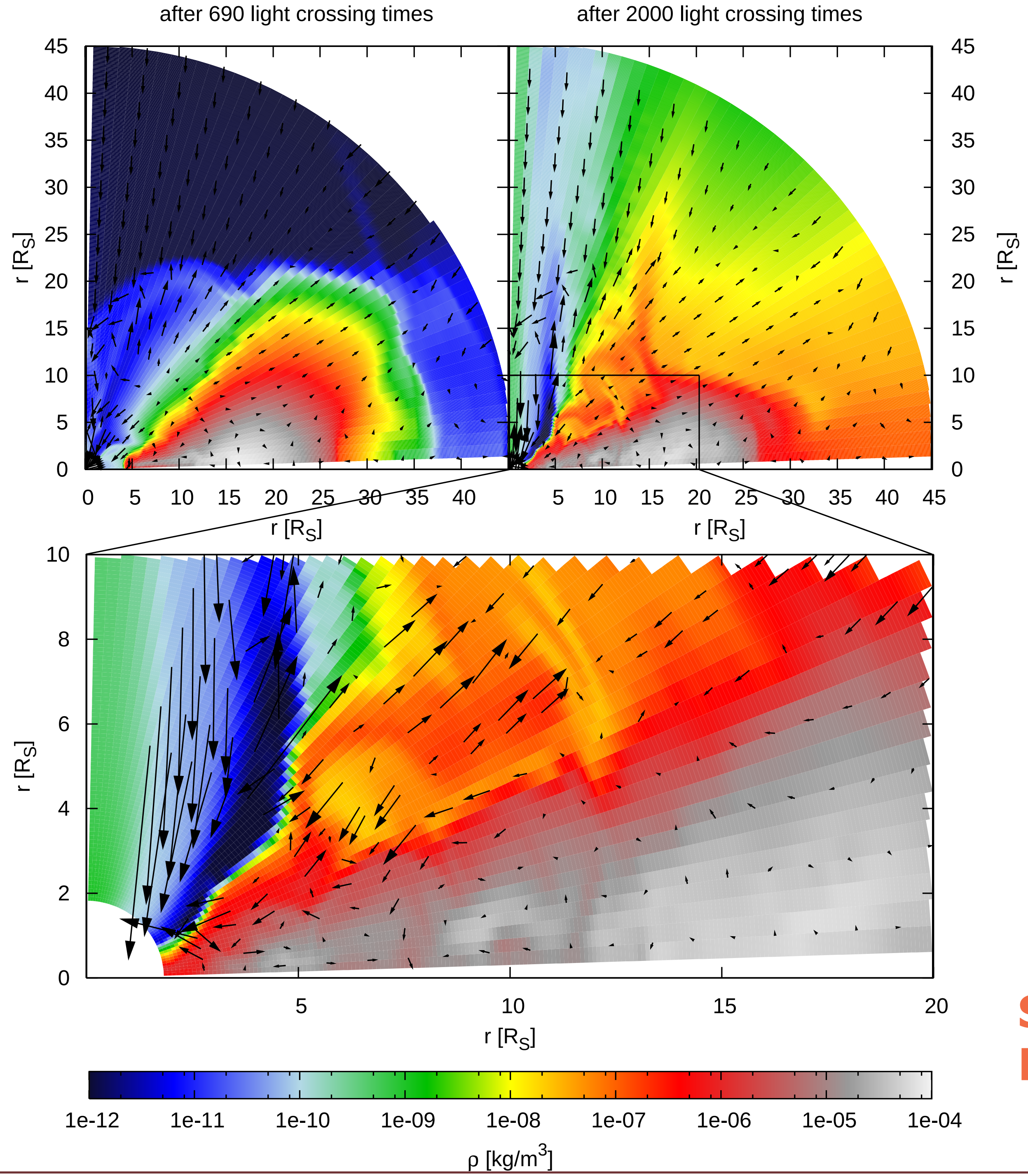
S. Brinkmann, LSW Heidelberg 2005



S. Brinkmann, LSW Heidelberg 2005

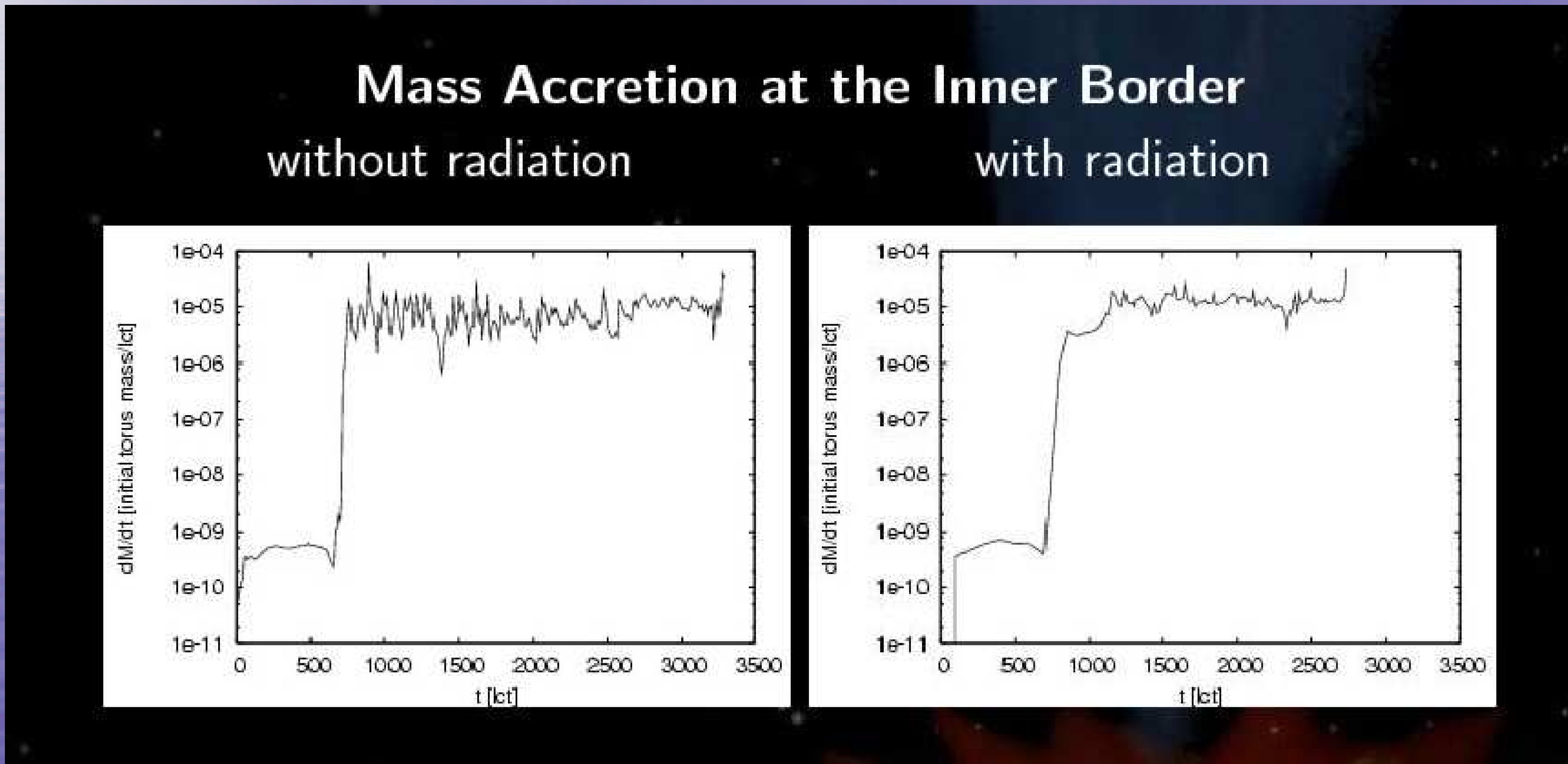


S. Brinkmann, LSW Heidelberg 2005



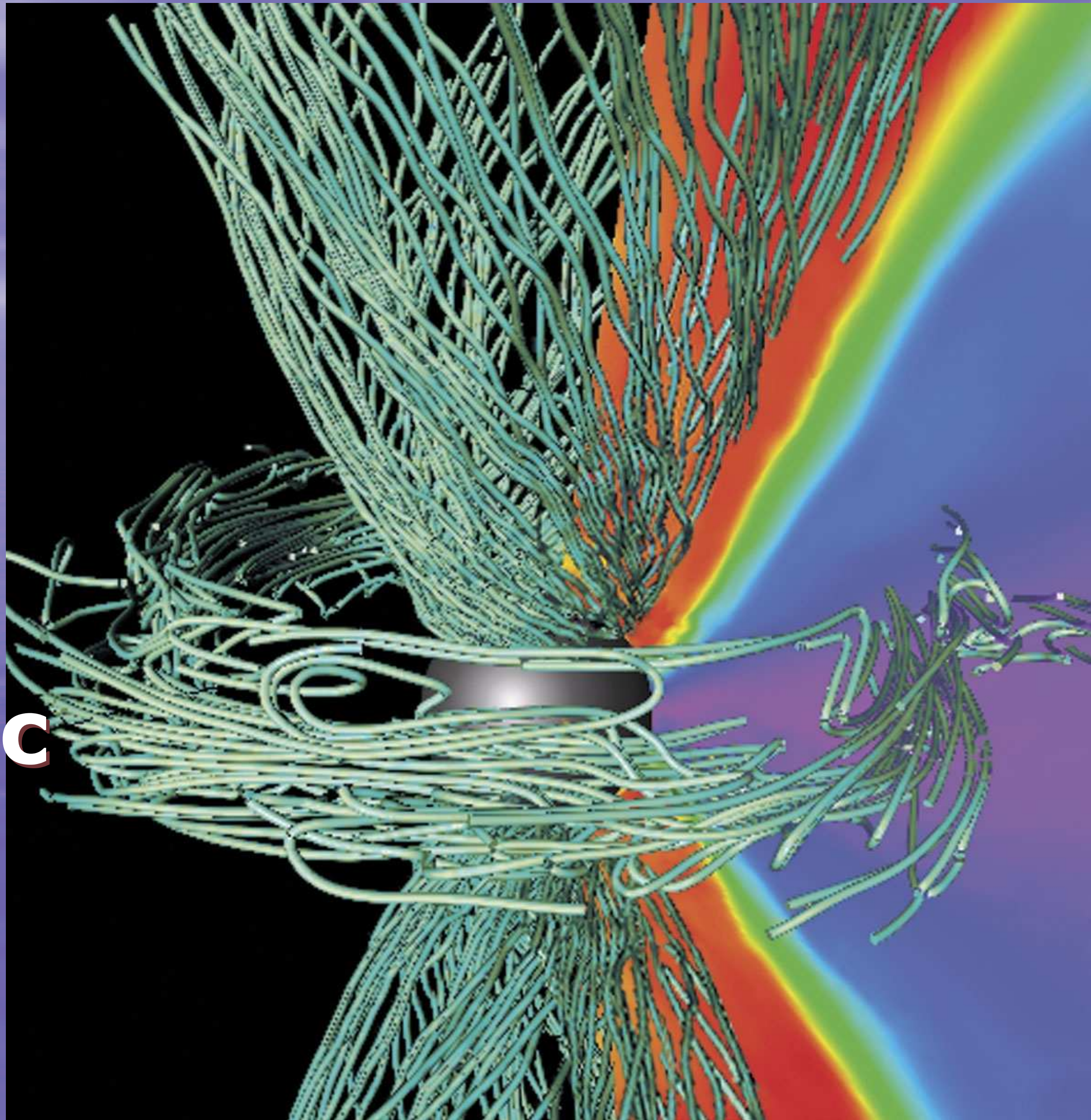
S. Brinkmann,
LSW Heidelberg 2005

Turbulence



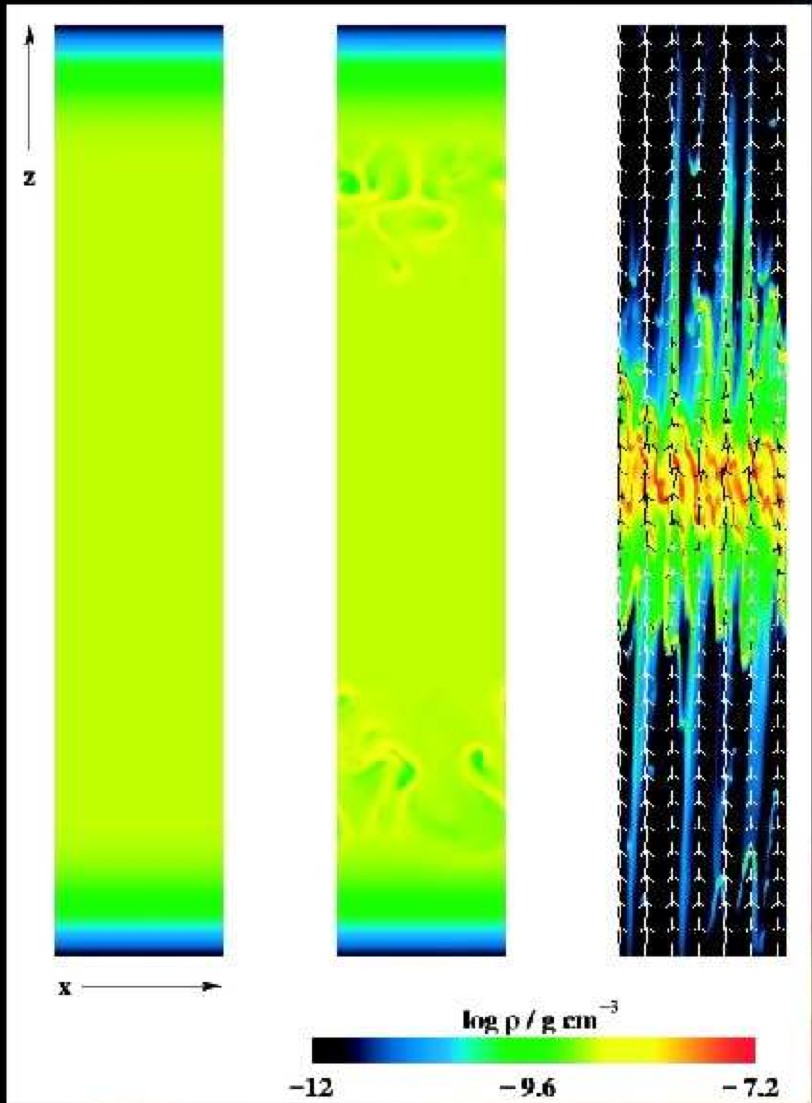
Mass Accretion at the Inner Border
without radiation
with radiation
1e-04
1e-05
1e-06
1e-07
1e-08
1e-09
1e-10
1e-11
0
500
1000
1500
2000
2500
3000
3500
t [lct]
dM/dt [initial torus mass/lct]

Outflows in Micro-Quasars → Relativistic Effects



Krolik 2005

Photon Bubbles and Shock Trains



Effect of Radiation →Collapse of disk

[Turner et al., 2005]

Beyond Plain MHD

Plasma effects

- Resistive MHD:
- Reduced 2D Hall (Grasso et al, 19
- Electron inertia and compressibili
- 3D Hall MHD and two-fluid MHD
- Radiative MHD
- Relativistic MHD

$$\mathbf{E}+\mathbf{V}\times\mathbf{B}=\frac{1}{S}\mathbf{J}+\frac{d_e^2}{n}\frac{d\mathbf{J}}{dt}+\frac{d_i}{n}\left(\mathbf{J}\times\mathbf{B}-\nabla\cdot\vec{\vec{p}}_e\right)$$